Homework assignment, Oct. 24, 2007.

1. Let Ω be an open convex subset of $\mathbb{R}^n$. A function $f : \Omega \to \mathbb{R}$ is said to be convex if for any two points $\mathbf{x}_2$, $\mathbf{x}_2 \in \Omega$ the inequality

$$f(t\mathbf{x}_1 + (1-t)\mathbf{x}_2) \leq tf(\mathbf{x}_1) + (1-t)f(\mathbf{x}_2)$$

holds for all $t \in [0, 1]$.

Prove that if $f \in C^2(\Omega)$ then f is convex if and only if $d^2 f_{\mathbf{x}}$ is positive semidefinite (i.e. $d^2 f_{\mathbf{x}}[\mathbf{h}] \geq 0$ for all $\mathbf{h}$) for all $x \in \Omega$.

You can use one-dimensional result from calculus.

2. For which values of α the function $f(x, y) = x^2/(1+y)^\alpha$ is convex in the domain $x \in \mathbb{R}$, $y > 0$?

To apply the theory you need to recall that the second differential is a quadratic form, and you should know from linear algebra how to check that a quadratic form is positive (semi)definite. If you do not recall, see "Linear Algebra Done Wrong", or any other linear algebra text.

3. Prove the Taylor's formula

$$f(\mathbf{x} + \mathbf{h}) = \sum_{k=0}^{N} \frac{d^k f_{\mathbf{x}}[\mathbf{h}]}{k!} + o(\mathbf{h})$$

by applying the Taylor's formula with the reminder

$$f(\mathbf{x} + \mathbf{h}) = \sum_{k=0}^{N} \frac{d^k f_{\mathbf{x}}[\mathbf{h}]}{k!} + \frac{d^{N+1} f_{\mathbf{x}+\theta\mathbf{h}}[\mathbf{h}]}{(N+1)!}$$

for $N - 1$.

4. Obtain from the Taylor's formula with differentials the Taylor's formula in multiindex notation, as in the text.

5. Analyze the proof of Theorem 8.24. Where the continuity of $D_i D_j$ is used? Why is the last formula in the proof "more than we need to obtain the desired conclusion"?

6. State and prove the sufficient condition of local minimum in terms of second differential.