Homework assignment, Oct. 31, 2007.

1. Find the formula for the inverse of the block matrix

$$\begin{pmatrix} I & 0 \\ B & C \end{pmatrix},$$

where I is the identity matrix (of size, say $n \times n$) and C is an invertible matrix ($m \times m$).

2. Prove that the set $SL(n)$ of all $n \times n$ matrices with determinant 1 is a manifold in the space $M_{n\times n}$ of all $n \times n$ matrices. What is its dimension.

Hint: you only need to show that equation $\det X = 1$ does not degenerate at $X = 1$, because multiplication by a matrix A^{-1}, $\det A = 1$ moves the point $A \in SL(n)$ to I, and the multiplication by A^{-1} maps bijectively $SL(n)$ onto itself.

3. Prove that the set $O(n)$ of orthogonal matrices (i.e. the set of real $n \times n$ matrices A satisfying $A^T A = I$ is a manifold. Find its dimension.

Hint: From the first glance it looks like the dimension should be zero, because in the equation $X^T X = I$ we have n^2 variables (entries of X) and n^2 equations. However some of these equations are duplicating each other.

Probably the easiest way to treat the duplication is to notice that the matrix $A^T A$ is always symmetric. So we can treat the function $f(X) = X^T X - I$ as a function acting from the space of $n \times n$ matrices to the space of $n \times n$ *symmetric* matrices (the latter space has the dimension smaller than n^2).

You should show that df_X has full rank. Again, it is enough to show only that df_I has full rank, see the hint to the previous problem.