# Homework assignment, Nov. 26, 2007.

1. Let $E \subset \mathbb{R}^n$. Prove that

a) If $H_p(E) < \infty$ then $H_r(E) = 0 \; \forall r > p$

b) If $H_p(E) > 0$ (the case $H_p(E) = \infty$ is also possible), then $H_r(E) = \infty \; \forall r < p$.

Use these fact to show that there exists a unique $p_0 \le n$ such that $H_p(E) = 0 \; \forall p > p_0$ and $H_p(E) = \infty \; \forall p < p_0$. Show that $p_0$ can be computed as

$$\begin{aligned} p_0 &= \sup\{p : H_p(E) > 0\} = \sup\{p : H_p(E) = \infty\} \\ &= \inf\{p : H_p(E) = 0\} = \inf\{p : H_p(E) < \infty\} \end{aligned}$$

2. Let $C$ be the 1/3 Cantor set. Show that for any $p > \ln 2/\ln 3$ we have $H_p(C) = 0$. That is the easier part of the proof that the Hausdorff dimension of the Cantor set is $\ln 2/\ln 3$.

Note, that the computation of the Hausdorff dimension of $C$ in class was a *heuristic* reasoning, not a rigorous proof. We assumed that for some $p$ $H_p(C)$ is different from 0 and $\infty$, and concluded that $p$ must be equal $\ln 2/\ln 3$ in this case. But that assumption needs to be proved, if fails for general sets, see the problem below.

3. Give an example of a *bounded* set $A \subset \mathbb{R}^2$ such that its Hausdorff dimension is 1 but $H_1(A) = \infty$.

4. Prove that the Hausdorff dimension of an interval $I = [a, b]$ is 1, and $H_1(I)$ equals the length of $I$.