Math. 1130 Fall 2007. Midterm 1, take home part solutions.

1. Prove that for the ball $B_{a,r} = \{x \in X : \rho(a,x) < r\}$, $r > 0$ in a metric space X

$$\operatorname{clos} B_{a,r} \subset \{x \in X : \rho(a,x) \le r\}.$$

Give an example, showing that the inclusion can be proper (i.e. not equality).

Proof. Map $f : X \to \mathbb{R}$, $f(x) = \rho(x,a)$ is continuous (see Problem 3). Therefore the sety

$$\{x \in X : \rho(x,a) \le r\} = f^{-1}([0,1])$$

is closed as an inverse image of a closed set. Thus $\operatorname{clos} B_{a,r} \subset \{x \in X : \rho(x,a) \le r\}$ because $\operatorname{clos} A$ is the intersection of all closed sets containing A.

To see that the inclusion is proper, consider $X = \mathbb{R} \setminus (0,1)$ (with metric inherited from $\mathbb{R}$, $\rho(x,y) = |x-y|$.

Then $B_{0,1} = (-10]$,

$$\{x \in X : \rho(x,a) \le 1\} = [-1,0] \cup \{1\}$$

and $\operatorname{clos} B_{0,1} = [-1,0]$ □

2. Prove that the sphere $S = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$ is not homeomorphic to the circle $\mathbb{T} = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| = 1\}$

Proof. Let $a = (0,0,1) \in S$, $b = (0,0,-1) \in S$ (south and north pole). If $f : S \to \mathbb{T}$ is a homeomorphism, then $f : S \setminus \{a\} \setminus \{b\} \to \mathbb{T} \setminus \{f(a)\} \setminus \{f(b)\}$ is also a homeomorphism.

Notice, that $\mathbb{T} \setminus \{f(a)\} \setminus \{f(b)\}$ is disconnected (because is is union of 2 disjoint open arcs, and an open arc is open in $\mathbb{T}$ ans so in $\mathbb{T} \setminus \{f(a)\} \setminus \{f(b)\}$).

On the other hand the set $S \setminus \{a\} \setminus \{b\}$ is path connected, and so connected. Indeed, for 2 points $x, y \in S \setminus \{a\} \setminus \{b\}$ we can construct a path connecting x and y by moving x along the "parallel" x is on (i.e. along the circle $\varphi = \text{const}$ in the spherical coordinates $z = \cos\varphi, x = \sin\varphi\cos\theta, y = \sin\varphi\sin\theta$) until it has the same "longitude" as y, and them moving it to y along the meridian (i.e. along the line $\theta = \text{const}$).

Since a continuous image of a connected set is connected, we got a contradiction. □

3. Let X be a metric space and $a \in X$. Show that the function $f : X \to \mathbb{R}$, $f(x) = \rho(x,a)$ is continuous. **Hint:** use triangle inequality.

Proof. By triangle inequality for all $x, y \in X$

$$\rho(x,a) \le \rho(y,a) + rho(x,y)$$

so

$$\rho(x,a) - \rho(y,a) \le rho(x,y).$$

Interchanging y and x we get

$$\rho(y, a) - \rho(x, a) \leq rho(x, y),$$

so combining the above 2 inequalities yield

$$|\rho(x, a) - \rho(y, a)| \leq rho(x, y).$$

Thus, for every $\varepsilon > 0$ the inequality $\rho(x, y) < \varepsilon$ implies

$$|\rho(y, a) - \rho(x, a)| \leq rho(x, y) < \varepsilon.$$

so f is even uniformly continuous. □

4. For a non-empty subset A of a metric space define the distance $d(x, A)$ from a point x to the set A as $d(x, A) := \inf\{\rho(x, y) : y \in A\}$.

Show that the function $x \mapsto d(x, A)$ is a continuous function on X.

Proof. Take arbitrary $\varepsilon > 0$. Let $x, y \in X$, $\rho(x, y) < \varepsilon/2$.

Since $d(x, A) := \inf\{\rho(x, y) : y \in A\}$, there exists $a \in A$ such that

$$\rho(x, a) < d(x, A) + \varepsilon/2$$

($d(x, A) + \varepsilon/2$ is not a lower bound for the st $\{\rho(x, y) : y \in A\}$).

Therefore, since $d(y, A) \leq \rho(y, a)$,

$$d(y, A) \leq \rho(y, a) \leq \rho(x, a) + \rho(x, y) < d(x, A) + \varepsilon/2 + \varepsilon/2 = d(x, A) + \varepsilon.$$

Repeating the same reasoning with x and y interchanged (there will be different a), we get

$$d(x, A) < d(y, A) + \varepsilon.$$

Therefore, for all $\varepsilon > 0$ and for all $x, y \in X$ such that $\rho(x, y) < \varepsilon/2$

$$|d(x, A) - d(y, A)| < \varepsilon,$$

so the function $x \mapsto d(x, A)$ is uniformly continuous (and so continuous). □

5. For an infinite (may be uncountable) family of non-negative numbers x_α, $\alpha \in \mathcal{A}$ one can define its sum $\sum_{\alpha \in \mathcal{A}} x_\alpha$ by considering all finite sums and taking their supremum.

Show, that if $\sum_{\alpha \in \mathcal{A}} x_\alpha < \infty$, then only countably many x_α are non-zero.

This shows that it does not make much sense to consider sums of uncountably many terms.

Hint: If the sum is finite, say S, how many terms can be greater than 1? Greater than 1/2?...

Proof. For $n \in \mathbb{N}$ define

$$A_n := \{\alpha \in \mathcal{A} : x_\alpha \geq n\}.$$

If

$$\sum_{\alpha \in \mathcal{A}} x_\alpha = S < \infty,$$

then clearly $\operatorname{card}(A_n) \leq nS$. Since

$$\{\alpha \in \mathcal{A} : x_\alpha > 0\} = \bigcup_{n \in \mathbb{N}} A_n,$$

the set $\{\alpha \in \mathcal{A} : x_\alpha > 0\}$ is countable as a countable union of countable (finite in our case) sets. $\square$

6. Prove the graph of the function $y = \sin(1/x)$, $x \in \mathbb{R} \setminus \{0\}$ in $\mathbb{R}^2$ together with the interval $x = 0$, $-1 \leq y \leq 1$ in $\mathbb{R}^2$ is connected but not path connected.

To show that the set is connected, one can notice that the set is a union of 3 obviously connected sets (which sets, and why are they connected?).

Proof. The whole set (cal it S) can be represented as a union of 3 disjoint sets,

$$S_1 = \{(x, y) \in \mathbb{R}^2 : y = \sin \frac{1}{x}, x \in (-\infty, 0)\} \tag{1}$$

$$S_2 = \{(0, y) \in \mathbb{R}^2 : y \in [-1, 1])\} \tag{2}$$

$$S_3 = \{(x, y) \in \mathbb{R}^2 : y = \sin \frac{1}{x}, x \in (0, \infty)\} \tag{3}$$

Let us show that S is connected.

The set S_2 is an interval, so it is connected. The sets S_1, S_3 are images of intervals $(-\infty, 0)$ and $(0, \infty)$ under the continuous map $F\mathbb{R} \setminus \{0\} \to \mathbb{R}^2$, $F(x) = (x, \sin(1/x)$, so they are connected as well.

Let A be a non-trivial open and closed subset of S, and let $B = S \setminus A$.

Notice, that $A \cap S_1$ is open and closed in S_1, so, since S_1 is connected, there are only 2 possibilities: either $A \cap S_1 = \varnothing$ or $A \cap S_1 = S_1$.

Similar reasoning applies for S_2 and S_3, so, foe each S_k, $k = 1, 2, 3$ there are only 2 possibilities: either $A \cap S_k = \varnothing$ or $A \cap S_k = S_1$.

So A can be either one of the sets S_k or a union of 2 of them (there are 6 possibilities all together).

It is easy to see that the sets S_2, $S_1 \cup S_2$ and $S_3 \cup S_2$ are not open in S. Therefore, the sets $S_1 \cup S_3$, S_3, S_1 are not closed in S.

These 6 sets cover all possible choices for a non-trivial A, and each of the sets is either not closed, or not open. So a non-trivial open and closed $A \subset S$ does not exist.

Another way to prove that S is connected is to use theorems from the textbook. Namely, $\operatorname{clos} S_1 = S_1 \cap S_2$, so $S_2 \cap S_2$ is connected as a closure of a connected set. Similarly, $\operatorname{clos} S_3 =$

$S_3 \cap S_2$, so $S_3 \cap S_2$ is connected. Therefore $S = (S_1 \cup S_2) \cup (S_3 \cup S_2)$ is connected as a union of connected sets with non-empty intersection.

Let us now prove that S is not path connected. Suppose there exists a path (function) $f : [0,1] \to S$ connecting points $(0,0)$ and $(1/\pi, 0)$, i.e. such that $f(0) = (0,0)$, $f(1) = (1/\pi.0)$.

Let f_1 f_2 be the coordinate functions of f, so $f = (f_1, f_2)$. The functions f_1, f_2 are clearly continuous.

Since $f_1(1) = 1/\pi$ By Intermediate Value Theorem there exists t_1, $0 < t_1 < t_0$ such that $f_1(t_1) = 1/(\pi/2 + \pi)$, so $f(t_1) = (1/(\pi/2 + \pi), -1)$.

Applying the Intermediate Value Theorem again we find $t_2 \in (0, t_1)$ such that $f_1(t_2) = 1/(\pi/2+2\pi)$. Repeating this process, we construct inductively a strictly decreasing sequence $\{t_n\}_1^\infty$, $t_n \in (0, 1/\pi)$, $t_{n+1} < t_n$, such that

$$f_1(t_n) = \frac{1}{\pi/2 + n\pi}$$

so $f_2(t_n) = (-1)^n$.

Since $\{t_n\}_1^\infty$ is a bounded monotone sequence, there exist $\lim_{n\to\infty} t_n =: t_0 \in [0,1]$.

By the continuity of f_2

$$\lim_{n\to\infty} f_2(t_n) = f_2(t_0).$$

But on the other hand, since $f_2(t_n) = (-1)^n$, the limit does not exist, so we got a contradiction. $\square$