#1 $f(\vec{x}) = \varphi(|\vec{x}|)$

$$\frac{\partial f}{\partial x_k} = \varphi'(|x|)\frac{\partial |\vec{x}|}{\partial x_k} = \varphi'(|x|)\cdot\frac{x_k}{|x|}$$

$$\frac{\partial}{\partial x_k}|\vec{x}| = \frac{\partial}{\partial x_k}\left(\sum x_j^2\right)^{1/2} = \frac{1}{2}\left(\sum x_j^2\right)^{-1/2}\cdot 2x_k$$
$$= \frac{x_k}{|\vec{x}|}$$

$$\frac{\partial^2 f}{\partial x_k^2} = \varphi''(|\vec{x}|)\frac{x_k^2}{|\vec{x}|^2} + \varphi'(|\vec{x}|)\frac{|\vec{x}|\cdot 1 - x_k\,|\vec{x}|^{-1}x_k}{|\vec{x}|^2}$$
$$= \varphi''(|x|)\frac{x_k}{|\vec{x}|^2} + \varphi'(|\vec{x}|)\left[\frac{1}{|\vec{x}|} - \frac{x_k^2}{|\vec{x}|^3}\right]$$

Summing over k we get

$$\Delta f = \sum_{k=1}^{n}\frac{\partial^2 f}{\partial x_k^2} = \varphi''(|\vec{x}|) + \varphi'(|\vec{x}|)\frac{n-1}{|\vec{x}|}.$$

So, $\Delta f = 0$ is equivalent to the differential equation,

$$\varphi''(r) + \frac{n-1}{r}\varphi'(r) = 0$$

Denoting $y = \varphi'$, we get

$$y' = -\frac{n-1}{r}y$$

$$\frac{dy}{dr} = -\frac{n-1}{r}y$$

$$\frac{dy}{y} = -(n-1)\frac{dr}{r}$$

$$\ln|y| = -(n-1)\ln|r| + C \qquad n \geq 2$$

$$y = C_1 r^{-(n-1)}$$

$$\varphi' = C_1 r^{-n+1}$$

$$\varphi = \frac{C_1}{-n+2} r^{-n+2} + C_2 = C_1 r^{-n+2} + C_2 \qquad n \geq 3$$

$$\varphi = C_1 \ln r + C_2 \qquad n = 2$$

So

$$f(\vec{x}) = \begin{cases} \dfrac{C_1}{|\vec{x}|^{n-2}} + C_2 & n \geq 3 \\ C_1 \ln|\vec{x}| + C_2 & n = 2 \end{cases}$$

So, for f to be defined at $\vec{x} = \vec{0}$ C_1 must be 0, so $f(\vec{x}) \equiv C_2$

Remark: If $n = 1$ $\Delta f = f''$ so harmonic functions are affine ones,

$$f(x) = ax + b$$

#2 Let $\vec{y}_0 \in f(\Omega)$, so $\vec{y}_0 = f(\vec{x}_0)$, $x_0 \in \Omega$

$f'(\vec{x}_0)$ is non-singular (i.e. invertible)

so by Inverse Function Theorem

$\exists$ neighborhood $U \ni x_0$ s.t. $f(U)$ is open

~~f~~ is injective on U and

inverse g of $f|_U$ $g: f(U) \to U$

is C^1

not essential here.

So y_0 is an interior point of $f(\Omega)$

(because $f(U) \subset f(\Omega)$)

Since y_0 is arbitrary we can

conclude that $f(\Omega)$ is open.

#3. There are countably many algebraic numbers (There are countably many polynomials with integer coefficients (can be proved by induction in the degree) and each polynomial has finitely many roots), so set of alg. #s is a countable union of countable (finite) sets)

Let $\{a_k\}_1^\infty$ be enumeration of the set of algebraic numbers. Take $\varepsilon > 0$.

Intervals $I_k = (a_k - \varepsilon 2^{-k}, a_k + \varepsilon 2^{-k})$ cover $A = \bigcup_1^\infty a_k$, and

$$\sum m(I_k) = \sum_{k=1}^{\infty} \varepsilon \cdot 2 \cdot 2^{-k} = 2\varepsilon$$

So $m^*(A) \le 2\varepsilon$

Because $\varepsilon > 0$ is arbitrary, we conclude that

$$m^*(A) = 0.$$

#4

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

Therefore $\forall n$ $\bigcup_{k=n}^{\infty} E_k \supset E$.

Thus,

$$\mu(E) \leq \sum_{k=n}^{\infty} \mu(E_k) \qquad \forall n$$

So

$$0 \leq \mu(E) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(E_k) = 0$$

Since $\sum_{1}^{\infty} \mu(E_k) < \infty$

#5

a) $f(x) = e^{x^2} + e^{y^2} + e^{z^2} + e^{w^2} - 8$

$f' = (2xe^{x^2}, 2ye^{y^2}, 2ze^{z^2}, 2we^{w^2})$

so f' has full rank $(=1)$ everywhere except $\vec{0}$, which does not belong to S. So, S is a manifold of dimension $4-1=3$

b) $S = (0,0,0,0)$

Manifold of dimension 0.

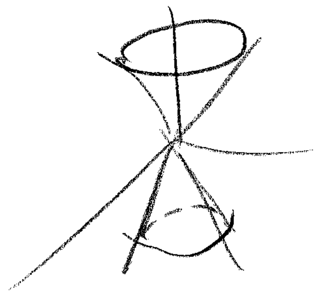
c) Defining function $f = x^2 + y^2 - z^2 - 1$

$f' = (2x, 2y, -2z)$

f' has full rank $(=1)$ everywhere except $\vec{0}$, and $\vec{0} \notin S$

so S is a manifold of dim $3-1=2$

d) $x^2+y^2-z^2=0$ defines a cone



If S were a manifold, it must have dim $2 = 3-1$ (1 — max possible rank of f')

So, by Implicit function thm in a neighborhood of each point of S, it can be represented as a graph of a functions of 2 variables (by chosing appropriate 2 variables as arguments)

But that is impossible in a neigh of $(0,0,0)$. (Why?)

e)

$$\Phi' = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 2\theta \end{pmatrix}$$

so Φ' always has rank 2

$$\det\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & \cos\theta \end{pmatrix} = r \neq 0$$

So S is a manifold of dimension 2