VECTORS, CURVES, AND SURFACES IN SPACE

12

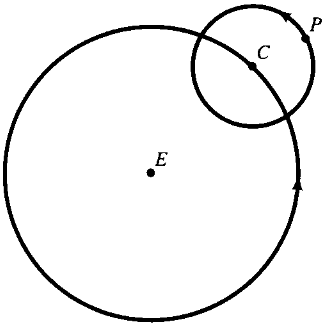
Johannes Kepler (1571–1630)

Ancient Greek mathematicians and astronomers developed an elaborate mathematical model to account for the complicated motions of the sun, moon, and six planets then known as viewed from the earth. A combination of uniform circular motions was used to describe the motion of each body around the earth—if the earth is arbitrarily placed at the origin of coordinates, then each body *does* orbit the earth.

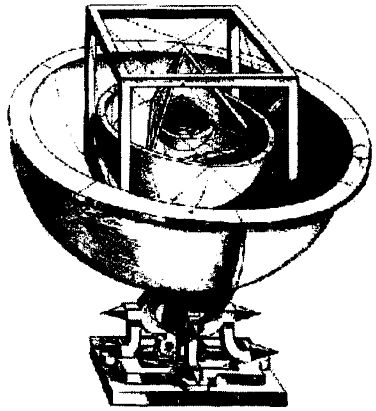
In this system, it was typical for a planet P to travel uniformly around a small circle (the *epicycle*) with center C, which in turn traveled uniformly around a circle centered at the earth (labeled E in the figure at the lower left). The radii of the circles and the angular speeds of P and C were chosen to match the observed motion of the planet as closely as possible. For greater accuracy, secondary "circles on circles" could be used. In fact, several circles were required for each body in the Greek theory of epicycles, which reached its definitive form in Ptolemy's *Almagest* of the second century A.D.

In 1543, Copernicus altered Ptolemy's approach by placing the center of each primary circle at the sun rather than at the earth. But this change was of greater philosophical than mathematical significance. His *heliocentric system* was still overly complicated, still requiring many secondary circles, and still beset with inaccuracies in representing the motions of the heavenly bodies.

It was Johannes Kepler who finally got rid of all these circles. On the basis of a detailed analysis of planetary observations accumulated by the Danish astronomer Tycho Brahe, Kepler stated his three famous *laws of planetary motion,* which describe elliptical (rather than circular) orbits of planets around the sun (Section 12.6). Ironically, his original goal had been to prove that the placement of Mercury, Venus, Earth, Mars, and Jupiter is determined by the five regular polyhedra as indicated in the figure at the lower right, which appeared in his *Mysterium Cosmographicum* (1596). This model of the solar system shows a cube inscribed in the sphere containing Saturn's orbit, and the sphere of Jupiter's orbit is inscribed in this cube. A tetrahedron (with four triangular faces) is inscribed in Jupiter's sphere, and in this tetrahedron is inscribed the sphere of the orbit of Mars. Continuing in this way, the spheres of the three remaining planets then known were interspersed with the remaining three regular solids—the octahedron (eight triangular faces), the dodecahedron (12 pentagonal faces), and the icosahedron (20 triangular faces). It is said that Kepler always remained prouder of his five solids than of his three laws.



The small circle is the epicycle.



Kepler's regular polyhedron model of the solar system.

In his *Principia Mathematica* (1687), Newton showed that Kepler's laws follow from the basic principles of mechanics ($F = ma$ and so on) and the inverse-square law of gravitational attraction. His success in using mathematics to explain natural phenomena ("I now demonstrate the frame of the System of the World") inspired confidence that the universe could be understood and perhaps even mastered. This new confidence permanently altered humanity's perception of itself and of its place in the scheme of things. Newton employed a powerful but now antiquated form of geometrical calculus in the *Principia*. In Section 12.6 we apply the modern calculus of vector-valued functions to outline the relation between Newton's laws and Kepler's laws.

12.1 VECTORS IN THE PLANE

A physical quantity such as length, temperature, or mass can be specified in terms of a single real number, its *magnitude*. Such a quantity is called a **scalar.** Other physical quantities, such as force and velocity, possess both magnitude and *direction;* these quantities are called **vector quantities,** or simply **vectors.**

For example, to specify the velocity of a moving point in the coordinate plane, we must give both the rate at which it moves (its speed) and the direction of that motion. The *velocity vector* of the moving point incorporates both pieces of information—direction and speed. It is convenient to represent this velocity vector by an arrow, with its initial point located at the current position of the moving point on its trajectory (Fig. 12.1.1).

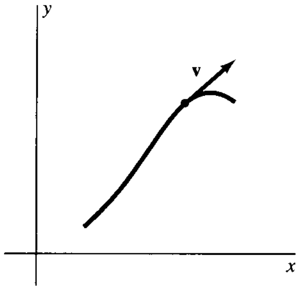


FIGURE 12.1.1 A velocity vector may be represented by an arrow.

Although the arrow, a directed line segment, carries the desired information—both magnitude (the segment's length) and direction—it is a pictorial representation rather than a quantitative object. The following formal definition of a vector captures the essence of magnitude in combination with direction.

DEFINITION Vector

A **vector** $\mathbf{v}$ in the Cartesian plane is an ordered pair of real numbers that has the form $\langle a, b\rangle$. We write $\mathbf{v} = \langle a, b\rangle$ and call a and b the **components** of the vector $\mathbf{v}$.

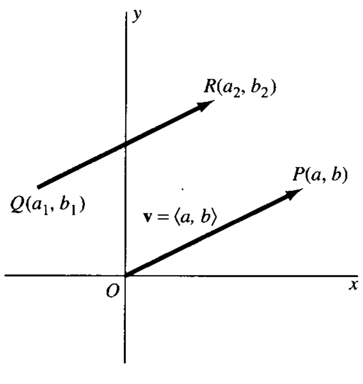


FIGURE 12.1.2 The position vector **v** of the point P and another representation $\overrightarrow{QR}$ of **v**.

The directed line segment $\overrightarrow{OP}$ from the origin O to the point $P(a, b)$ is one geometric representation of the vector $\mathbf{v}$. (See Fig. 12.1.2.) For this reason, the vector $\mathbf{v} = \langle a, b\rangle$ is called the **position vector** of the point $P(a, b)$. In fact, the relationship between $\mathbf{v} = \langle a, b\rangle$ and $P(a, b)$ is so close that, in certain contexts, it is convenient to confuse the two deliberately—to regard $\mathbf{v}$ and P as the same mathematical object.

The directed line segment from the point $Q(a_1, b_1)$ to the point $R(a_2, b_2)$ has the same direction and magnitude as the directed line segment from the origin $O(0, 0)$ to the point $P(a, b)$ with $a = a_2 - a_1$ and $b = b_2 - b_1$ (Fig. 12.1.2), and consequently they represent the same vector $\mathbf{v} = \overrightarrow{OP} = \overrightarrow{QR}$. This observation makes it easy to find the components of the vector with arbitrary initial point Q and arbitrary terminal point R.

REMARK When discussing vectors we often use the term *scalar* to refer to an ordinary numerical quantity, one that is *not* a vector. In printed work we use **bold** type to distinguish the names of vectors from those of other mathematical objects, such as the scalars a and b that are the components of the vector $\mathbf{v} = \langle a, b\rangle$. In handwritten work a suitable alternative is to place an arrow—or just a bar—over every symbol that denotes a vector. Thus you may write $\vec{\mathrm{v}} = \langle \mathrm{a}, \mathrm{b}\rangle$ or $\bar{\mathrm{v}} = \langle \mathrm{a}, \mathrm{b}\rangle$. There is no need for an arrow or a bar over a vector $\langle a, b\rangle$ already identified by angle brackets, so none should be used there.

A directed line segment has both length and direction. The **length** of the vector $\mathbf{v} = \langle a, b\rangle$ is denoted by $v = |\mathbf{v}|$ and is defined as follows:

$$v = |\mathbf{v}| = |\langle a, b\rangle| = \sqrt{a^2 + b^2}. \qquad \textbf{(1)}$$

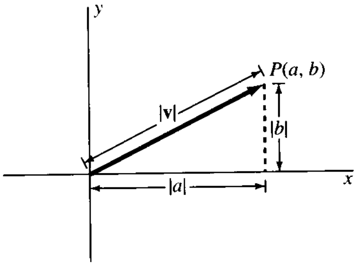


FIGURE 12.1.3 The length $v = |\mathbf{v}|$ of the vector **v**.

The notation $v = |\mathbf{v}|$ is used because the length of a vector is in many ways analogous to the absolute value of a real number (Fig. 12.1.3).

EXAMPLE 1 The length of the vector $\mathbf{v} = \langle 1, -2 \rangle$ is

$$v = |\langle 1, -2 \rangle| = \sqrt{(1)^2 + (-2)^2} = \sqrt{5}.$$

◆

The only vector with length zero is the **zero vector** with both components zero, denoted by $\mathbf{0} = \langle 0, 0 \rangle$. The zero vector is unique in that it has no specific direction. Every nonzero vector has a specified direction; the vector represented by the arrow $\overrightarrow{OP}$ from the origin O to another point P in the plane has direction specified (for instance) by the counterclockwise angle from the positive x-axis to $\overrightarrow{OP}$.

What is important about the vector $\mathbf{v} = \langle a, b \rangle$ represented by $\overrightarrow{OP}$ often is not *where* it is, but how long it is and which way it points. If the directed line segment $\overrightarrow{QR}$ with endpoints $Q(a_1, b_1)$ and $R(a_2, b_2)$ has the same length and direction as $\overrightarrow{OP}$, then we say that $\overrightarrow{QR}$ **represents** (or is a **representation** of) the vector **v** (Fig. 12.1.2). Thus a single vector has many representatives (Fig. 12.1.4).

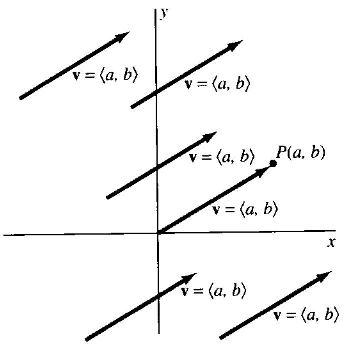


FIGURE 12.1.4 All these arrows represent the same vector $\mathbf{v} = \langle a, b \rangle$.

Algebraic Operations with Vectors

The operations of addition and multiplication of real numbers have analogues for vectors. We shall define each of these operations of *vector algebra* in terms of components of vectors and then give a geometric interpretation in terms of arrows.

DEFINITION Equality of Vectors

The two vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are **equal** provided that $u_1 = v_1$ and $u_2 = v_2$.

In other words, two vectors are equal if and only if *corresponding components* are the same. Moreover, two directed line segments $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ represent the same vector provided that they have the same length and direction. This will be the case provided that the segments $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are opposite sides of a parallelogram (Fig. 12.1.5).

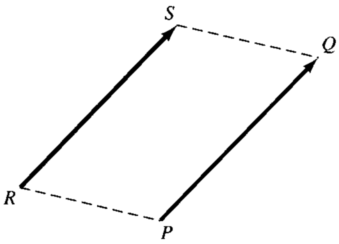


FIGURE 12.1.5 Parallel directed segments representing equal vectors.

DEFINITION Addition of Vectors

The **sum** $\mathbf{u} + \mathbf{v}$ of the two vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is the vector

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle. \qquad \mathbf{(2)}$$

Thus we add vectors by adding corresponding components—that is, by *componentwise addition.* The geometric interpretation of vector addition is the **triangle law of addition,** illustrated in Fig. 12.1.6, where the labeled lengths indicate why this interpretation is valid. An equivalent interpretation is the **parallelogram law of addition,** illustrated in Fig. 12.1.7.

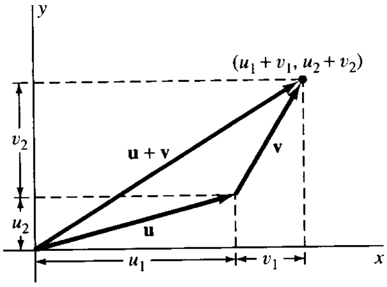


FIGURE 12.1.6 The triangle law is a geometric interpretation of vector addition.

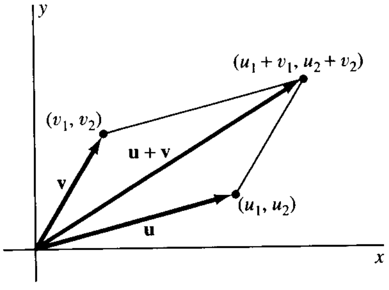


FIGURE 12.1.7 The parallelogram law for vector addition.

EXAMPLE 2 The sum of the vectors $\mathbf{u} = \langle 4, 3\rangle$ and $\mathbf{v} = \langle -5, 2\rangle$ is the vector

$$\mathbf{u} + \mathbf{v} = \langle 4, 3\rangle + \langle -5, 2\rangle = \langle 4 + (-5), 3 + 2\rangle = \langle -1, 5\rangle. \qquad \blacklozenge$$

It is natural to write $2\mathbf{u} = \mathbf{u} + \mathbf{u}$. But if $\mathbf{u} = \langle u_1, u_2\rangle$, then

$$2\mathbf{u} = \mathbf{u} + \mathbf{u} = \langle u_1, u_2\rangle + \langle u_1, u_2\rangle = \langle 2u_1, 2u_2\rangle.$$

This suggests that multiplication of a vector by a scalar (real number) also is defined in a componentwise manner.

DEFINITION Multiplication of a Vector by a Scalar

If $\mathbf{u} = \langle u_1, u_2\rangle$ and c is a real number, then the **scalar multiple** $c\mathbf{u}$ is the vector

$$c\mathbf{u} = \langle cu_1, cu_2\rangle. \qquad \textbf{(3)}$$

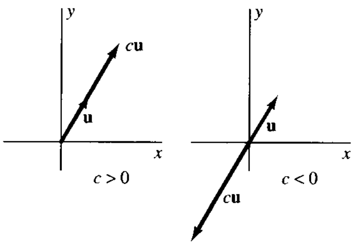


FIGURE 12.1.8 The vector $c\mathbf{u}$ may have the same direction as $\mathbf{u}$ or the opposite direction, depending on the sign of c.

Note that

$$|c\mathbf{u}| = \sqrt{(cu_1)^2 + (cu_2)^2} = |c|\sqrt{(u_1)^2 + (u_2)^2} = |c| \cdot |\mathbf{u}|.$$

Thus the length of $|c\mathbf{u}|$ is $|c|$ times the length of $\mathbf{u}$. The **negative** of the vector $\mathbf{u}$ is the vector

$$-\mathbf{u} = (-1)\mathbf{u} = \langle -u_1, -u_2\rangle,$$

with the same length as $\mathbf{u}$ but the opposite direction. We say that the two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ have

- The **same direction** if $\mathbf{u} = c\mathbf{v}$ for some $c > 0$;
- **Opposite directions** if $\mathbf{u} = c\mathbf{v}$ for some $c < 0$.

The geometric interpretation of scalar multiplication is that $c\mathbf{u}$ is the vector with length $|c| \cdot |\mathbf{u}|$, with the same direction as $\mathbf{u}$ if $c > 0$ but with the opposite direction if $c < 0$ (Fig. 12.1.8).

The **difference** $\mathbf{u} - \mathbf{v}$ of the vectors $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ is defined to be

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \langle u_1 - v_1, u_2 - v_2\rangle. \qquad \textbf{(4)}$$

If we think of $\langle u_1, u_2\rangle$ and $\langle v_1, v_2\rangle$ as position vectors of the points P and Q, respectively, then $\mathbf{u} - \mathbf{v}$ may be represented by the arrow $\overrightarrow{QP}$ from Q to P. We may therefore write

$$\mathbf{u} - \mathbf{v} = \overrightarrow{OP} - \overrightarrow{OQ} = \overrightarrow{QP},$$

as illustrated in Fig. 12.1.9.

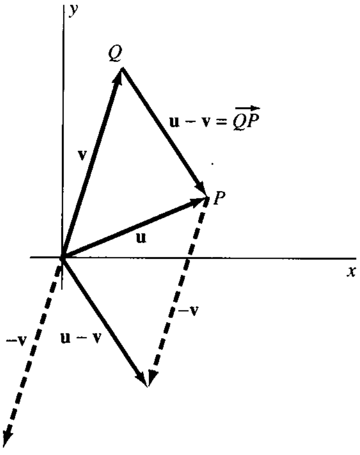


FIGURE 12.1.9 Geometric interpretation of the difference $\mathbf{u} - \mathbf{v}$.

EXAMPLE 3 Suppose that $\mathbf{u} = \langle 4, -3\rangle$ and $\mathbf{v} = \langle -2, 3\rangle$. Find $|\mathbf{u}|$ and the vectors $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, $3\mathbf{u}$, $-2\mathbf{v}$, and $2\mathbf{u} + 4\mathbf{v}$.

Solution

$$|\mathbf{u}| = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5.$$

$$\mathbf{u} + \mathbf{v} = \langle 4 + (-2), -3 + 3\rangle = \langle 2, 0\rangle.$$

$$\mathbf{u} - \mathbf{v} = \langle 4 - (-2), -3 - 3\rangle = \langle 6, -6\rangle.$$

$$3\mathbf{u} = \langle 3 \cdot 4, 3 \cdot (-3)\rangle = \langle 12, -9\rangle.$$

$$-2\mathbf{v} = \langle -2 \cdot (-2), -2 \cdot 3\rangle = \langle 4, -6\rangle.$$

$$2\mathbf{u} + 4\mathbf{v} = \langle 2 \cdot 4 + 4 \cdot (-2), 2 \cdot (-3) + 4 \cdot 3\rangle = \langle 0, 6\rangle. \qquad \blacklozenge$$

The familiar algebraic properties of real numbers carry over to the following analogous properties of vector addition and scalar multiplication. Let **a**, **b**, and **c** be vectors and r and s real numbers. Then

$$\begin{aligned} &\textbf{1.}\quad \mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a},\\ &\textbf{2.}\quad \mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c},\\ &\textbf{3.}\quad r(\mathbf{a}+\mathbf{b})=r\mathbf{a}+r\mathbf{b},\\ &\textbf{4.}\quad (r+s)\mathbf{a}=r\mathbf{a}+s\mathbf{a},\\ &\textbf{5.}\quad (rs)\mathbf{a}=r(s\mathbf{a})=s(r\mathbf{a}). \end{aligned} \tag{5}$$

You can easily verify these identities by working with components. For example, if $\mathbf{a}=\langle a_1, a_2\rangle$ and $\mathbf{b}=\langle b_1, b_2\rangle$, then

$$\begin{aligned} r(\mathbf{a}+\mathbf{b}) &= r\langle a_1+b_1,\ a_2+b_2\rangle = \langle r(a_1+b_1),\ r(a_2+b_2)\rangle \\ &= \langle ra_1+rb_1,\ ra_2+rb_2\rangle = \langle ra_1,\ ra_2\rangle + \langle rb_1,\ rb_2\rangle = r\mathbf{a}+r\mathbf{b}. \end{aligned}$$

The proofs of the other four identities in (5) are left as exercises.

The Unit Vectors i and j

A **unit** vector is a vector of length 1. If $\mathbf{a}=\langle a_1, a_2\rangle \neq \mathbf{0}$, then

$$\mathbf{u}=\frac{\mathbf{a}}{|\mathbf{a}|} \tag{6}$$

is the unit vector with the same direction as **a**, because

$$|\mathbf{u}| = \sqrt{\left(\frac{a_1}{|\mathbf{a}|}\right)^2 + \left(\frac{a_2}{|\mathbf{a}|}\right)^2} = \frac{1}{|\mathbf{a}|}\sqrt{a_1^2+a_2^2} = 1.$$

For example, if $\mathbf{a}=\langle 3, -4\rangle$, then $|\mathbf{a}|=5$. Thus $\langle \frac{3}{5}, -\frac{4}{5}\rangle$ is a unit vector that has the same direction as **a**.

Two particular unit vectors play a special role, the vectors

$$\mathbf{i}=\langle 1, 0\rangle \quad \text{and} \quad \mathbf{j}=\langle 0, 1\rangle.$$

The first points in the positive x-direction; the second points in the positive y-direction (Fig. 12.1.10). Together they provide a useful alternative notation for vectors. If $\mathbf{a}=\langle a_1, a_2\rangle$, then

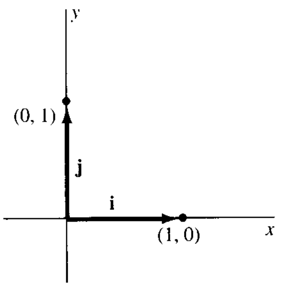


FIGURE 12.1.10 The vectors **i** and **j**.

$$\mathbf{a}=\langle a_1, 0\rangle + \langle 0, a_2\rangle = a_1\langle 1, 0\rangle + a_2\langle 0, 1\rangle = a_1\mathbf{i}+a_2\mathbf{j}. \tag{7}$$

Thus every vector in the plane is a **linear combination** of **i** and **j**. The usefulness of this notation is based on the fact that such linear combinations of **i** and **j** may be manipulated as if they were ordinary sums. For example, if

$$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j} \quad \text{and} \quad \mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j},$$

then

$$\mathbf{a}+\mathbf{b}=(a_1\mathbf{i}+a_2\mathbf{j})+(b_1\mathbf{i}+b_2\mathbf{j})=(a_1+b_1)\mathbf{i}+(a_2+b_2)\mathbf{j}.$$

Also,

$$c\mathbf{a}=c(a_1\mathbf{i}+a_2\mathbf{j})=(ca_1)\mathbf{i}+(ca_2)\mathbf{j}.$$

EXAMPLE 4 Suppose that $\mathbf{a}=2\mathbf{i}-3\mathbf{j}$ and $\mathbf{b}=3\mathbf{i}+4\mathbf{j}$. Express $5\mathbf{a}-3\mathbf{b}$ in terms of **i** and **j**.

Solution

$$\begin{aligned} 5\mathbf{a}-3\mathbf{b} &= 5\cdot(2\mathbf{i}-3\mathbf{j})-3\cdot(3\mathbf{i}+4\mathbf{j}) \\ &= (10-9)\mathbf{i}+(-15-12)\mathbf{j}=\mathbf{i}-27\mathbf{j}. \end{aligned}$$

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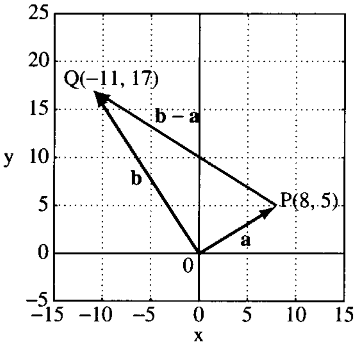


FIGURE 12.1.11 The vectors **a**, **b**, and **b** − **a** of Example 5.

EXAMPLE 5 When the vectors $\mathbf{a} = 8\mathbf{i} + 5\mathbf{j}$ and $\mathbf{b} = -11\mathbf{i} + 17\mathbf{j}$ are plotted carefully (Fig. 12.1.11), they look as though they might be perpendicular. Determine whether or not this is so.

Solution If the vectors **a** and **b** are regarded as position vectors of the points $P(8, 5)$ and $Q(-11, 17)$, then their difference $\mathbf{c} = \mathbf{b} - \mathbf{a} = -19\mathbf{i} + 12\mathbf{j}$ represents the third side $\overrightarrow{PQ}$ of the triangle OPQ (Fig. 12.1.11). According to the Pythagorean theorem, this triangle is a right triangle with hypotenuse PQ if and only if $|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2$. But

$$|\mathbf{c}|^2 = (-19)^2 + 12^2 = 505 \quad \text{whereas} \quad |\mathbf{a}|^2 + |\mathbf{b}|^2 = [8^2 + 5^2] + [(-11)^2 + 17^2] = 499.$$

It follows that the vectors **a** and **b** are not perpendicular. ◆

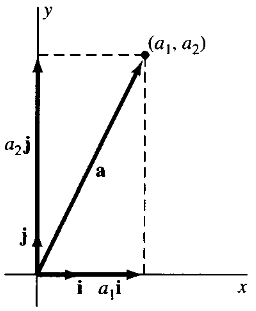


FIGURE 12.1.12 Resolution of $\mathbf{a} = \langle a_1, a_2 \rangle$ into its horizontal and vertical components.

Equation (7) expresses the vector $\mathbf{a} = \langle a_1, a_2 \rangle$ as the sum of a horizontal vector $a_1\mathbf{i}$ and a vertical vector $a_2\mathbf{j}$, as Fig. 12.1.12 shows. The decomposition or *resolution* of a vector into its horizontal and vertical components is an important technique in the study of vector quantities. For example, a force **F** may be decomposed into its horizontal and vertical components $F_1\mathbf{i}$ and $F_2\mathbf{j}$, respectively. The physical effect of the single force **F** is the same as the combined effect of the separate forces $F_1\mathbf{i}$ and $F_2\mathbf{j}$. (This is an instance of the empirically verifiable parallelogram law of addition of forces.) Because of this decomposition, many two-dimensional problems can be reduced to one-dimensional problems, the latter solved, and the two results combined (again by vector methods) to give the solution of the original problem.

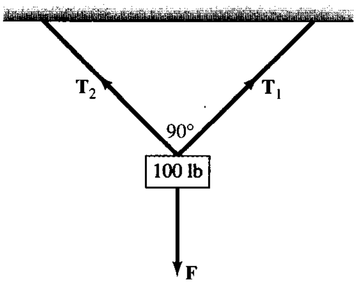


FIGURE 12.1.13 The suspended weight of Example 6.

EXAMPLE 6 A 100-lb weight is suspended from the ceiling by means of two perpendicular flexible cables of equal length (Fig. 12.1.13). Find the tension (in pounds) in each cable.

Solution Each cable is inclined at an angle of 45° from the horizontal, so it follows readily upon calculating horizontal and vertical components that the indicated tension force vectors $\mathbf{T}_1$ and $\mathbf{T}_2$ are given by

$$\mathbf{T}_1 = (T_1 \cos 45°)\mathbf{i} + (T_1 \sin 45°)\mathbf{j} \quad \text{and} \quad \mathbf{T}_2 = (-T_2 \cos 45°)\mathbf{i} + (T_2 \sin 45°)\mathbf{j},$$

where $T_1 = |\mathbf{T}_1|$ and $T_2 = |\mathbf{T}_2|$ are the tension forces we seek. The downward force of gravity acting on the weight is given by $\mathbf{F} = -100\mathbf{j}$. In order that the weight hangs motionless, the three forces must "balance," so that $\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{F} = \mathbf{0}$; that is,

$$[(T_1 \cos 45°)\mathbf{i} + (T_1 \sin 45°)\mathbf{j}] + [(-T_2 \cos 45°)\mathbf{i} + (T_2 \sin 45°)\mathbf{j}] = 100\mathbf{j}.$$

When we equate the components of **i** in this equation and separately equate the components of **j**, we get the two scalar equations

$$T_1 \cos 45° - T_2 \cos 45° = 0 \quad \text{and} \quad T_1 \sin 45° + T_2 \sin 45° = 100.$$

The first of these scalar equations implies that $T_1 = T_2 = T$, and then the second yields $T = 100/(2 \sin 45°) = 50\sqrt{2} \approx 70.71$ (pounds) for the tension in each cable. ◆

12.1 TRUE/FALSE STUDY GUIDE

12.1 CONCEPTS: QUESTIONS AND DISCUSSION

1. Discuss the relation between a 2-dimensional vector and a point in the plane.
2. Give several examples of quantities that possess both magnitude and direction. For each, discuss whether and how such quantities might be added.
3. If a person owns stock in two companies, how might the worth of his portfolio be described by a 2-dimensional vector? If several people owning stock in these same two companies form a partnership, is the "worth vector" of the partnership equal to the sum of the worth vectors of the partners?

12.1 PROBLEMS

In Problems 1 through 4, find a vector $\mathbf{v} = \langle a, b\rangle$ *that is represented by the directed line segment* $\overrightarrow{RS}$*. Then sketch both* $\overrightarrow{RS}$ *and the position vector of the point* $P(a, b)$*.*

1. $R(1, 2)$, $S(3, 5)$ **2.** $R(-2, -3)$, $S(1, 4)$

3. $R(5, 10)$, $S(-5, -10)$ **4.** $R(-10, 20)$, $S(15, -25)$

In Problems 5 through 8, find the sum $\mathbf{w} = \mathbf{u} + \mathbf{v}$ *and illustrate it geometrically.*

5. $\mathbf{u} = \langle 1, -2\rangle$, $\mathbf{v} = \langle 3, 4\rangle$ **6.** $\mathbf{u} = \langle 4, 2\rangle$, $\mathbf{v} = \langle -2, 5\rangle$

7. $\mathbf{u} = 3\mathbf{i} + 5\mathbf{j}$, $\mathbf{v} = 2\mathbf{i} - 7\mathbf{j}$ **8.** $\mathbf{u} = 7\mathbf{i} + 5\mathbf{j}$, $\mathbf{v} = -10\mathbf{i}$

In Problems 9 through 16, find $|\mathbf{a}|$, $|-2\mathbf{b}|$, $|\mathbf{a} - \mathbf{b}|$, $\mathbf{a} + \mathbf{b}$, *and* $3\mathbf{a} - 2\mathbf{b}$.

9. $\mathbf{a} = \langle 1, -2\rangle$, $\mathbf{b} = \langle -3, 2\rangle$

10. $\mathbf{a} = \langle 3, 4\rangle$, $\mathbf{b} = \langle -4, 3\rangle$

11. $\mathbf{a} = \langle -2, -2\rangle$, $\mathbf{b} = \langle -3, -4\rangle$

12. $\mathbf{a} = -2\langle 4, 7\rangle$, $\mathbf{b} = -3\langle -4, -2\rangle$

13. $\mathbf{a} = \mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = 2\mathbf{i} - 5\mathbf{j}$

14. $\mathbf{a} = 2\mathbf{i} - 5\mathbf{j}$, $\mathbf{b} = \mathbf{i} - 6\mathbf{j}$

15. $\mathbf{a} = 4\mathbf{i}$, $\mathbf{b} = -7\mathbf{j}$

16. $\mathbf{a} = -\mathbf{i} - \mathbf{j}$, $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j}$

In Problems 17 through 20, find a unit vector $\mathbf{u}$ *with the same direction as the given vector* $\mathbf{a}$*. Express* $\mathbf{u}$ *in terms of* $\mathbf{i}$ *and* $\mathbf{j}$*. Also find a unit vector* $\mathbf{v}$ *with the direction opposite that of* $\mathbf{a}$*.*

17. $\mathbf{a} = \langle -3, -4\rangle$ **18.** $\mathbf{a} = \langle 5, -12\rangle$

19. $\mathbf{a} = 8\mathbf{i} + 15\mathbf{j}$ **20.** $\mathbf{a} = 7\mathbf{i} - 24\mathbf{j}$

In Problems 21 through 24, find the vector $\mathbf{a}$*, expressed in terms of* $\mathbf{i}$ *and* $\mathbf{j}$*, that is represented by the arrow* $\overrightarrow{PQ}$ *in the plane.*

21. $P = (3, 2)$, $Q = (3, -2)$

22. $P = (-3, 5)$, $Q = (-3, 6)$

23. $P = (-4, 7)$, $Q = (4, -7)$

24. $P = (1, -1)$, $Q = (-4, -1)$

In Problems 25 through 28, determine whether or not the given vectors $\mathbf{a}$ *and* $\mathbf{b}$ *are perpendicular.*

25. $\mathbf{a} = \langle 6, 0\rangle$, $\mathbf{b} = \langle 0, -7\rangle$

26. $\mathbf{a} = 3\mathbf{j}$, $\mathbf{b} = 3\mathbf{i} - \mathbf{j}$

27. $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{b} = 4\mathbf{j} + 8\mathbf{i}$

28. $\mathbf{a} = 8\mathbf{i} + 10\mathbf{j}$, $\mathbf{b} = 15\mathbf{i} - 12\mathbf{j}$

In Problems 29 and 30, express $\mathbf{i}$ *and* $\mathbf{j}$ *in terms of* $\mathbf{a}$ *and* $\mathbf{b}$*.*

29. $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j}$

30. $\mathbf{a} = 5\mathbf{i} - 9\mathbf{j}$, $\mathbf{b} = 4\mathbf{i} - 7\mathbf{j}$

In Problems 31 and 32, write $\mathbf{c}$ *in the form* $r\mathbf{a} + s\mathbf{b}$ *where* r *and* s *are scalars.*

31. $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} - \mathbf{j}$, $\mathbf{c} = 2\mathbf{i} - 3\mathbf{j}$

32. $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = 8\mathbf{i} + 5\mathbf{j}$, $\mathbf{c} = 7\mathbf{i} + 9\mathbf{j}$

33. Find a vector that has the same direction as $5\mathbf{i} - 7\mathbf{j}$ and is (a) three times its length; (b) one-third its length.

34. Find a vector that has the opposite direction from $-3\mathbf{i} + 5\mathbf{j}$ and is (a) four times its length; (b) one-fourth its length.

35. Find a vector of length 5 with (a) the same direction as $7\mathbf{i} - 3\mathbf{j}$; (b) the direction opposite that of $8\mathbf{i} + 5\mathbf{j}$.

36. For what numbers c are the vectors $\langle c, 2\rangle$ and $\langle c, -8\rangle$ perpendicular?

37. For what numbers c are the vectors $2c\mathbf{i} - 4\mathbf{j}$ and $3\mathbf{i} + c\mathbf{j}$ perpendicular?

38. Given the three points $A(2, 3)$, $B(-5, 7)$, and $C(1, -5)$, verify by direct computation of the vectors and their sum that $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$.

In Problems 39 through 42, give a componentwise proof of the indicated property of vector algebra. Take $\mathbf{a} = \langle a_1, a_2\rangle$, $\mathbf{b} = \langle b_1, b_2\rangle$, *and* $\mathbf{c} = \langle c_1, c_2\rangle$ *throughout.*

39. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

40. $(r + s)\mathbf{a} = r\mathbf{a} + s\mathbf{a}$

41. $(rs)\mathbf{a} = r(s\mathbf{a})$

42. If $\mathbf{a} + \mathbf{b} = \mathbf{a}$, then $\mathbf{b} = \mathbf{0}$.

43. Find the tension in each cable of Example 6 if the angle between them is 120°.

In Problems 44 through 46, a given weight (in pounds) is suspended by two cables as shown in the figure. Find the tension in each cable.

44.

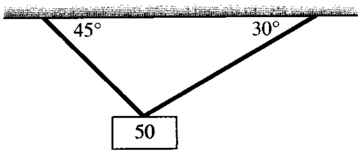


FIGURE 12.1.14

45.

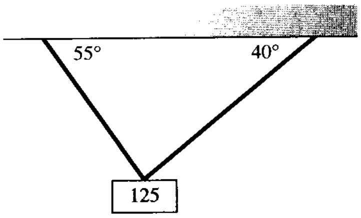


FIGURE 12.1.15

46.

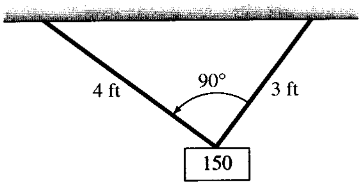


FIGURE 12.1.16

In Problems 47 through 49, assume the following fact: If an airplane flies with velocity vector $\mathbf{v}_a$ *relative to the air and the velocity of the wind is* $\mathbf{w}$, *then the velocity vector of the plane relative to the ground is* $\mathbf{v}_g = \mathbf{v}_a + \mathbf{w}$ *(Fig. 12.1.17). The vector* $\mathbf{v}_a$ *is called the* apparent velocity vector *and the vector* $\mathbf{v}_g$ *is called the* true velocity vector.

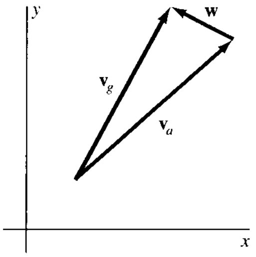


FIGURE 12.1.17 The vectors of Problems 47 through 49

- Apparent velocity: $\mathbf{v}_a$
- Wind velocity: $\mathbf{w}$
- True velocity: $\mathbf{v}_g = \mathbf{v}_a + \mathbf{w}$

47. Suppose that the wind is blowing from the northeast at 50 mi/h and that the pilot wishes to fly due east at 500 mi/h. What should the plane's apparent velocity vector be?

48. Repeat Problem 47 with the phrase *due east* replaced with *due west.*

49. Repeat Problem 47 in the case that the pilot wishes to fly northwest at 500 mi/h.

50. Given any three points A, B, and C in the plane, show that $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$. [*Suggestion:* Picture the triangle ABC.]

51. If $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of the points P and Q in the plane and M is the point with position vector $\mathbf{v} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$, show that M is the midpoint of the line segment PQ. Is it sufficient to show that the vectors $\overrightarrow{PM}$ and $\overrightarrow{QM}$ are equal and opposite?

52. In the triangle ABC, let M and N be the midpoints of AB and AC, respectively. Show that $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{BC}$. Conclude that the line segment joining the midpoints of two sides of a triangle is parallel to the third side. How are their lengths related?

53. Prove that the diagonals of a parallelogram $ABCD$ bisect each other. [*Suggestion:* If M and N are the midpoints of the diagonals AC and BD, respectively, and O is the origin, show that $\overrightarrow{OM} = \overrightarrow{ON}$.]

54. Use vectors to prove that the midpoints of the four sides of an arbitrary quadrilateral are the vertices of a parallelogram.

55. Figure 12.1.18 shows the vector $\mathbf{a}_\perp$ obtained by rotating the vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ through a counterclockwise angle of 90°. Show that

$$\mathbf{a}_\perp = -a_2\mathbf{i} + a_1\mathbf{j}.$$

[*Suggestion:* Begin by writing $\mathbf{a} = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j}$.]

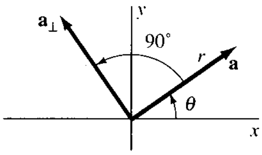


FIGURE 12.1.18 Rotate **a** counterclockwise 90° to obtain $\mathbf{a}_\perp$ (Problem 55).

12.2 THREE-DIMENSIONAL VECTORS

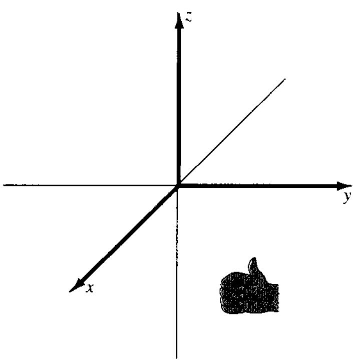
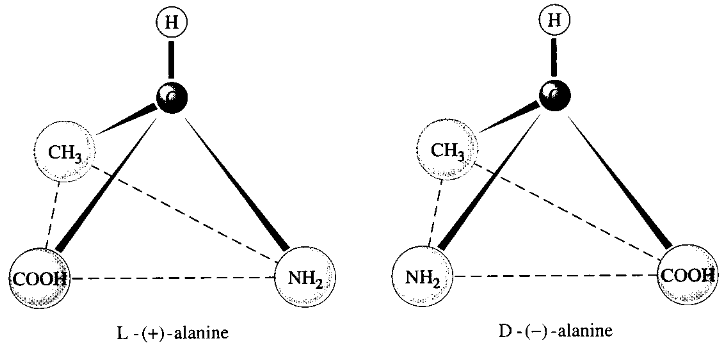


FIGURE 12.2.1 The right-handed coordinate system.

In the first eleven chapters we discussed many aspects of the calculus of functions of a *single* variable. The geometry of such functions is two-dimensional, because the graph of a function of a single variable is a curve in the coordinate plane. Most of the remaining chapters deal with the calculus of functions of *several* (two or more) independent variables. The geometry of functions of two variables is three-dimensional because the graphs of such functions are generally surfaces in space.

Rectangular coordinates in the plane may be generalized to rectangular coordinates in space. A point in space is determined by giving its location relative to three mutually perpendicular **coordinate axes** that pass through the origin O. We shall usually draw the x-, y-, and z-axes as shown in Fig. 12.2.1, sometimes with arrows indicating the positive direction along each axis; the positive x-axis will always be labeled x, and similarly for the positive y- and z-axes. With this configuration of axes, our rectangular coordinate system is said to be **right-handed:** If you curl the fingers of your right hand in the direction of a 90° rotation from the positive x-axis to the positive y-axis, then your thumb points in the direction of the positive z-axis. If the x- and y-axes were interchanged, then the coordinate system would be left-handed. These two coordinate systems are different in that it is impossible to bring one into coincidence with the other by means of rotations and translations. This is why the L- and D-alanine molecules shown in Fig. 12.2.2 are different; you can metabolize the left-handed ("levo") version but not the right-handed ("dextro") version. In this book we shall discuss right-handed coordinate systems exclusively and always draw the x-, y-, and z-axes with the right-handed orientation shown in Fig. 12.2.1.



L-(+)-alanine D-(−)-alanine

FIGURE 12.2.2 The stereoisomers of the amino acid alanine are physically and biologically different even through they have the same molecular formula.

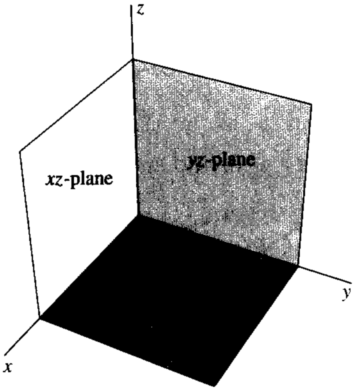


FIGURE 12.2.3 The coordinate planes in space.

The three coordinate axes taken in pairs determine the three **coordinate planes** (Fig. 12.2.3):

- The (horizontal) xy-plane, where $z = 0$;
- The (vertical) yz-plane, where $x = 0$; and
- The (vertical) xz-plane, where $y = 0$.

The point P in space is said to have **rectangular coordinates** (x, y, z) if

- x is its signed distance from the yz-plane,
- y is its signed distance from the xz-plane, and
- z is its signed distance from the xy-plane.

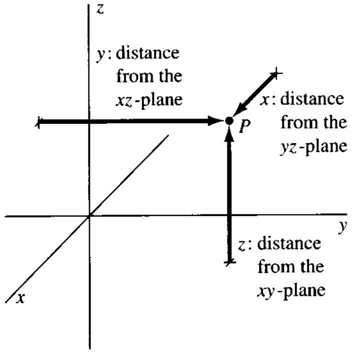


FIGURE 12.2.4 Locating the point P in rectangular coordinates.

(See Fig. 12.2.4.) In this case we may describe the location of P simply by calling it "the point $P(x, y, z)$." There is a natural one-to-one correspondence between ordered triples (x, y, z) of real numbers and points P in space; this correspondence is called a **rectangular coordinate system** in space. In Fig. 12.2.5 the point P is located in the **first octant**—the eighth of space in which all three rectangular coordinates are positive.

If we apply the Pythagorean theorem to the right triangles P_1QR and P_1RP_2 in Fig. 12.2.6, we get

$$\begin{aligned}|P_1P_2|^2 &= |RP_2|^2 + |P_1R|^2 = |RP_2|^2 + |QR|^2 + |P_1Q|^2\\ &= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2.\end{aligned}$$

Thus the **distance formula** for the **distance** $|P_1P_2|$ between the points P_1 and P_2 is

$$|P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}. \tag{1}$$

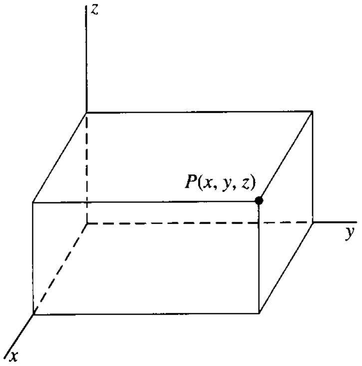


FIGURE 12.2.5 Completing the box to show P with the illusion of the third dimension.

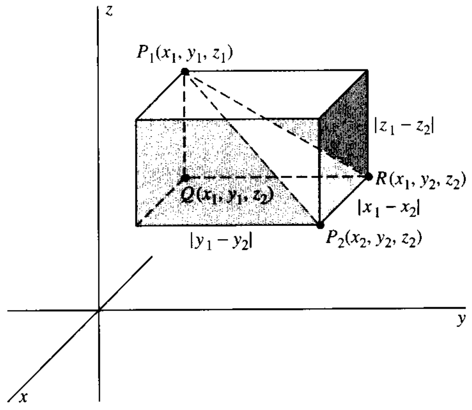


FIGURE 12.2.6 The distance between P_1 and P_2 is the length of the long diagonal of the box.

EXAMPLE 1 The distance between the points $A(1, 3, -2)$ and $B(4, -3, 1)$ is

$$|AB| = \sqrt{(4-1)^2 + (-3-3)^2 + (1+2)^2} = \sqrt{54} \approx 7.348.$$

◆

You can apply the distance formula in Eq. (1) to show that the **midpoint** M of the line segment joining $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right). \tag{2}$$

(See Problem 63.)

The **graph** of an equation in three variables x, y, and z is the set of all points in space with rectangular coordinates that satisfy that equation. In general, the graph of an equation in three variables is a *two-dimensional surface* in $\boldsymbol{R}^3$ (three-dimensional space with rectangular coordinates).

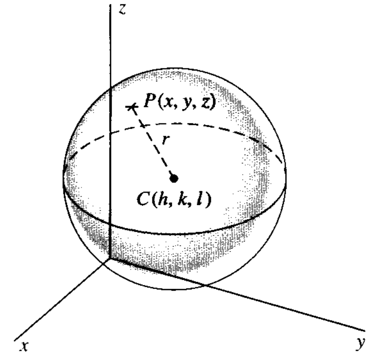


FIGURE 12.2.7 The sphere with center (h, k, l) and radius r.

EXAMPLE 2 Given a fixed point $C(h, k, l)$ and a number $r > 0$, find an equation of the sphere with radius r and center C.

Solution By definition, the sphere is the set of all points $P(x, y, z)$ such that the distance from P to C is r. That is, $|CP| = r$, and thus $|CP|^2 = r^2$. Therefore

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2. \tag{3}$$

◆

Equation (3) is worth remembering as the equation of the **sphere with radius** r and **center** $C(h, k, l)$ shown in Fig. 12.2.7. Moreover, given an equation of the form

$$x^2 + y^2 + z^2 + Ax + By + Cz + D = 0,$$

we can attempt—by completing the square in each variable—to write it in the form of Eq. (3) and thereby show that its graph is a sphere.

EXAMPLE 3 Determine the graph of the equation

$$x^2 + y^2 + z^2 + 4x + 2y - 6z - 2 = 0.$$

Solution We complete the square in each variable. The equation then takes the form

$$(x^2 + 4x + 4) + (y^2 + 2y + 1) + (z^2 - 6z + 9) = 2 + (4 + 1 + 9) = 16;$$

that is,

$$(x+2)^2 + (y+1)^2 + (z-3)^2 = 4^2.$$

Thus the graph of the given equation is the sphere with radius 4 and center $(-2, -1, 3)$. ◆

Vectors in Space

The discussion of vectors in space parallels the discussion in Section 12.1 of vectors in the plane. The difference is that a vector in space has three components rather than two. The point $P(x, y, z)$ has **position vector** $\mathbf{v} = \overrightarrow{OP} = \langle x, y, z\rangle$, which is represented by the directed line segment (or arrow) $\overrightarrow{OP}$ from the origin O to the point P (as well as by any parallel translate of this arrow—see Fig. 12.2.8). The distance formula in Eq. (1) gives

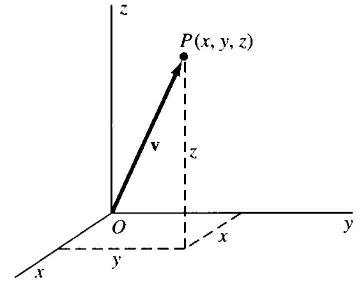


FIGURE 12.2.8 The arrow $\overrightarrow{OP}$ represents the position vector $\mathbf{v} = \langle x, y, z\rangle$.

$$|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2} \tag{4}$$

for the **length** (or **magnitude**) of the vector $\mathbf{v} = \langle x, y, z\rangle$.

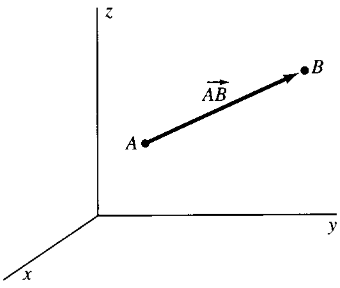


FIGURE 12.2.9 The arrow $\overrightarrow{AB}$ represents the vector $\mathbf{v} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$.

Given two points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ in space, the directed line segment $\overrightarrow{AB}$ in Fig. 12.2.9 represents the vector

$$\mathbf{v} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle.$$

Its length is the distance between the two points A and B:

$$|\mathbf{v}| = |\overrightarrow{AB}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}.$$

What it means for two vectors in space to be equal is essentially the same as in the case of two-dimensional vectors: The vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ are **equal** provided that $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$. That is, two vectors are equal exactly when corresponding components are equal.

We define addition and scalar multiplication of three-dimensional vectors exactly as we did in Section 12.1, taking into account that the vectors now have three components rather than two: The **sum** of the vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is the vector

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle. \tag{5}$$

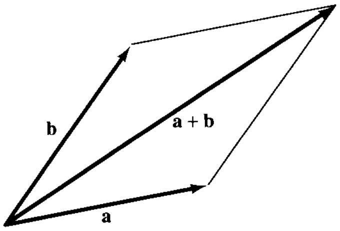


FIGURE 12.2.10 The parallelogram law for addition of vectors.

Because $\mathbf{a}$ and $\mathbf{b}$ lie in a plane (although not necessarily the xy-plane) if their initial points coincide, addition of three-dimensional vectors obeys the same **parallelogram law** as in the two-dimensional case (Fig. 12.2.10).

If c is a real number, then the **scalar multiple** $c\mathbf{a}$ is the vector

$$c\mathbf{a} = \langle ca_1, ca_2, ca_3 \rangle. \tag{6}$$

The length of $c\mathbf{a}$ is $|c|$ times the length of $\mathbf{a}$, and $c\mathbf{a}$ has the same direction as $\mathbf{a}$ if $c > 0$ but the opposite direction if $c < 0$. The following algebraic properties of vector addition and scalar multiplication for three-dimensional vectors are easy to establish; they follow from computations with components, exactly as in Section 12.1:

$$\begin{aligned}
\mathbf{a} + \mathbf{b} &= \mathbf{b} + \mathbf{a}, \\
\mathbf{a} + (\mathbf{b} + \mathbf{c}) &= (\mathbf{a} + \mathbf{b}) + \mathbf{c}, \\
r(\mathbf{a} + \mathbf{b}) &= r\mathbf{a} + r\mathbf{b}, \\
(r + s)\mathbf{a} &= r\mathbf{a} + s\mathbf{a}, \\
(rs)\mathbf{a} &= r(s\mathbf{a}) = s(r\mathbf{a}).
\end{aligned} \tag{7}$$

EXAMPLE 4 If $\mathbf{a} = \langle 3, 4, 12 \rangle$ and $\mathbf{b} = \langle -4, 3, 0 \rangle$, then

$$\begin{aligned}
\mathbf{a} + \mathbf{b} &= \langle 3 - 4, 4 + 3, 12 + 0 \rangle = \langle -1, 7, 12 \rangle, \\
|\mathbf{a}| &= \sqrt{3^2 + 4^2 + 12^2} = \sqrt{169} = 13, \\
2\mathbf{a} &= \langle 2 \cdot 3, 2 \cdot 4, 2 \cdot 12 \rangle = \langle 6, 8, 24 \rangle, \quad \text{and} \\
2\mathbf{a} - 3\mathbf{b} &= \langle 6 + 12, 8 - 9, 24 - 0 \rangle = \langle 18, -1, 24 \rangle.
\end{aligned}$$

◆

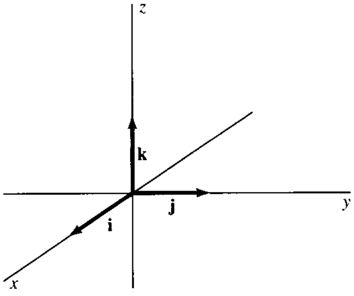


FIGURE 12.2.11 The basic unit vectors **i**, **j**, and **k**.

A **unit vector** is a vector of length 1. We can express any vector in space (or *space vector*) in terms of the three **basic unit vectors**

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \qquad \mathbf{j} = \langle 0, 1, 0 \rangle, \qquad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

When located with their initial points at the origin, these basic unit vectors form a right-handed triple of vectors pointing in the positive directions along the three coordinate axes (Fig. 12.2.11).

The space vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ can be written as

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$

a linear combination of the basic unit vectors. As in the two-dimensional case, the usefulness of this representation is that algebraic operations involving vectors may be carried out simply by collecting coefficients of **i**, **j**, and **k**.

EXAMPLE 5 Given the vectors $\mathbf{a} = \langle 3, -4, 2\rangle$ and $\mathbf{b} = \langle 5, 2, -7\rangle$, we can write

$$\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \mathbf{b} = 5\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}$$

in order to calculate

$$\begin{aligned} 7\mathbf{a} + 5\mathbf{b} &= 7 \cdot (3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) + 5 \cdot (5\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}) \\ &= (21 + 25)\mathbf{i} + (-28 + 10)\mathbf{j} + (14 - 35)\mathbf{k} \\ &= 46\mathbf{i} - 18\mathbf{j} - 21\mathbf{k} = \langle 46, -18, -21\rangle. \end{aligned}$$

◆

The Dot Product of Two Vectors

The **dot product** of the two vectors

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \text{and} \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

is the number obtained when we multiply corresponding components of **a** and **b** and add the results. That is,

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3. \tag{8}$$

Thus the dot product of two vectors is the *sum of the products of their corresponding components.* In the case of plane vectors $\mathbf{a} = \langle a_1, a_2\rangle$ and $\mathbf{b} = \langle b_1, b_2\rangle$, we simply dispense with third components and write $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2$.

EXAMPLE 6 To apply the definition to calculate the dot product of the two vectors $\mathbf{a} = \langle 3, 4, 12\rangle$ and $\mathbf{b} = \langle -4, 3, 0\rangle$, we simply follow the pattern in Eq. (8):

$$\mathbf{a} \cdot \mathbf{b} = (3)(-4) + (4)(3) + (12)(0) = -12 + 12 + 0 = 0.$$

And if $\mathbf{c} = \langle 4, 5, -3\rangle$, then

$$\mathbf{a} \cdot \mathbf{c} = (3)(4) + (4)(5) + (12)(-3) = 12 + 20 - 36 = -4.$$

◆

IMPORTANT The dot product of two *vectors* is a *scalar*—that is, an ordinary real number. For this reason the dot product is often called the **scalar product.** Example 6 illustrates the fact that the scalar product of two nonzero vectors (with positive lengths) may be zero or even a negative number.

The following **properties of the dot product** show that dot products of vectors behave in many ways in analogy to the ordinary algebra of real numbers.

$$\begin{aligned} \mathbf{a} \cdot \mathbf{a} &= |\mathbf{a}|^2, \\ \mathbf{a} \cdot \mathbf{b} &= \mathbf{b} \cdot \mathbf{a}, \\ \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}, \\ (r\mathbf{a}) \cdot \mathbf{b} &= r(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (r\mathbf{b}). \end{aligned} \tag{9}$$

Each of the properties in (9) can be established by working with components of the vectors involved. For instance, to establish the second equation, suppose that $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$. Then

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = b_1a_1 + b_2a_2 + b_3a_3 = \mathbf{b} \cdot \mathbf{a}.$$

This derivation makes it clear that the commutative law for the dot product is a consequence of the commutative law $ab = ba$ for multiplication of ordinary real numbers.

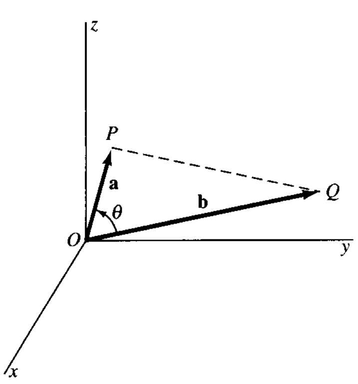


FIGURE 12.2.12 The angle θ between the vectors **a** and **b**.

Example 6 shows that the *algebraic definition* of the dot product is easy to apply in routine calculations. But what does it mean? The significance and **meaning of the dot product** lie in its *geometric interpretation.*

Let the vectors **a** and **b** be represented as position vectors by the directed segments $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, respectively. Then the angle θ between **a** and **b** is the angle at O in triangle OPQ of Fig. 12.2.12. We say that **a** and **b** are **parallel** if $\theta = 0$ or if $\theta = \pi$ and that **a** and **b** are **perpendicular** if $\theta = \pi/2$. For convenience, we regard the zero vector $\mathbf{0} = \langle 0, 0, 0\rangle$ as both parallel to *and* perpendicular to *every* vector.

THEOREM 1 Interpretation of the Dot Product

If θ is the angle between the vectors **a** and **b**, then

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta. \tag{10}$$

PROOF If either $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$, then Eq. (10) follows immediately. If the vectors **a** and **b** are parallel, then $\mathbf{b} = t\mathbf{a}$ with either $t > 0$ and $\theta = 0$ or $t < 0$ and $\theta = \pi$. In either case, both sides in Eq. (10) reduce to $t|\mathbf{a}|^2$, so again the conclusion of Theorem 1 follows.

We turn to the general case in which the vector $\mathbf{a} = \overrightarrow{OP}$ and $\mathbf{b} = \overrightarrow{OQ}$ are nonzero and nonparallel. Then

$$\begin{aligned}|\overrightarrow{QP}|^2 &= |\mathbf{a}-\mathbf{b}|^2 = (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b}\\ &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a}\cdot\mathbf{b}.\end{aligned}$$

But $c = |\overrightarrow{QP}|$ is the side of triangle OPQ (Fig. 12.2.12) that is opposite the angle θ included between the sides $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$. Hence the law of cosines (Appendix M) gives

$$\begin{aligned}|\overrightarrow{QP}|^2 &= c^2 = a^2 + b^2 - 2ab\cos\theta\\ &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\,|\mathbf{a}|\,|\mathbf{b}|\cos\theta.\end{aligned}$$

Finally, comparing these two expressions for $|\overrightarrow{QP}|^2$ yields Eq. (10). ◄

This theorem tells us that the angle θ between the nonzero vectors **a** and **b** can be found by using the equation

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}. \tag{11}$$

For instance, given the vectors $\mathbf{a} = \langle 8, 5\rangle$ and $\mathbf{b} = \langle -11, 17\rangle$ of Example 5 in Section 12.1, we calculate

$$\cos\theta = \frac{\langle 8, 5\rangle\cdot\langle -11, 17\rangle}{|\langle 8, 5\rangle|\,|\langle -11, 17\rangle|} = \frac{(8)(-11) + (5)(17)}{\sqrt{8^2+5^2}\sqrt{(-11)^2+17^2}} = \frac{-3}{\sqrt{89}\sqrt{410}}.$$

It follows that $\theta = \arccos(-3/\sqrt{89}\sqrt{410}) \approx 1.5865$ (*radians*) $\approx 90.90^\circ \neq 90^\circ$, so we see again that the vectors **a** and **b** are not perpendicular.

More generally, the two nonzero vectors **a** and **b** are perpendicular if and only if they make a right angle, so that $\theta = \pi/2$. By (11), this in turn is so if and only if $\mathbf{a}\cdot\mathbf{b} = 0$. Hence we have a quick computational check for perpendicularity of vectors.

COROLLARY Test for Perpendicular Vectors

The two nonzero vectors **a** and **b** are perpendicular if and only if $\mathbf{a}\cdot\mathbf{b} = 0$.

EXAMPLE 7 (a) To show that the plane vectors $\mathbf{a} = \langle 8, 5\rangle$ and $\mathbf{b} = \langle -11, 17\rangle$ of Example 5 in Section 12.1 were not perpendicular, we need only have calculated their dot product $\mathbf{a}\cdot\mathbf{b} = -88 + 85 = -3$ and observed that its value is not zero.
(b) Given the space vectors $\mathbf{a} = \langle 8, 5, -1\rangle$ and $\mathbf{b} = \langle -11, 17, -3\rangle$, we find that

$$\mathbf{a}\cdot\mathbf{b} = (8)(-11) + (5)(17) + (-1)(-3) = -88 + 85 + 3 = 0.$$

We may therefore conclude that **a** and **b** *are* perpendicular.

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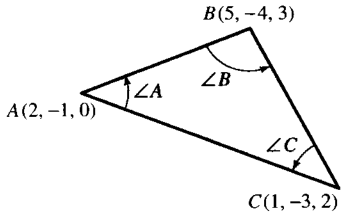

FIGURE 12.2.13 The triangle of Example 8.

EXAMPLE 8 Find the angles shown in the triangle of Fig. 12.2.13 with vertices at $A(2, -1, 0)$, $B(5, -4, 3)$, and $C(1, -3, 2)$.

Solution We apply Eq. (10) with $\theta = \angle A$, $\mathbf{a} = \overrightarrow{AB} = \langle 3, -3, 3\rangle$, and $\mathbf{b} = \overrightarrow{AC} = \langle -1, -2, 2\rangle$. This yields

$$\angle A = \cos^{-1}\left(\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}|\,|\overrightarrow{AC}|}\right) = \cos^{-1}\left(\frac{\langle 3, -3, 3\rangle\cdot\langle -1, -2, 2\rangle}{\sqrt{27}\sqrt{9}}\right)$$

$$= \cos^{-1}\left(\frac{9}{\sqrt{27}\,\sqrt{9}}\right) \approx 0.9553\,(\text{rad}) \approx 54.74^\circ.$$

Similarly,

$$\angle B = \cos^{-1}\left(\frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}|\,|\overrightarrow{BC}|}\right) = \cos^{-1}\left(\frac{\langle -3, 3, -3\rangle\cdot\langle -4, 1, -1\rangle}{\sqrt{27}\sqrt{18}}\right)$$

$$= \cos^{-1}\left(\frac{18}{\sqrt{27}\sqrt{18}}\right) \approx 0.6155\,(\text{rad}) \approx 35.26^\circ.$$

Then $\angle C = 180^\circ - \angle A - \angle B \approx 90^\circ$. As a check, note that

$$\overrightarrow{CA}\cdot\overrightarrow{CB} = \langle 1, 2, -2\rangle\cdot\langle 4, -1, 1\rangle = 0.$$

So the angle at C is, indeed, a right angle.

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Direction Angles and Projections

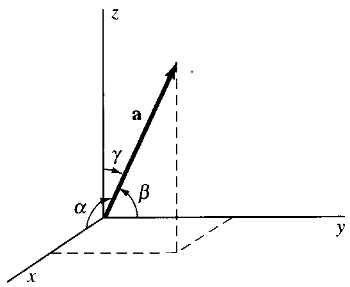

FIGURE 12.2.14 The direction angles of the vector **a**.

The **direction angles** of the nonzero vector $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ are the angles α, β, and γ that it makes with the vectors **i**, **j**, and **k**, respectively (Fig. 12.2.14). The cosines of these angles, $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, are called the **direction cosines** of the vector **a**. When we replace **b** in Eq. (11) with **i**, **j**, and **k** in turn, we find that

$$\cos\alpha = \frac{\mathbf{a}\cdot\mathbf{i}}{|\mathbf{a}|\,|\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|},$$

$$\cos\beta = \frac{\mathbf{a}\cdot\mathbf{j}}{|\mathbf{a}|\,|\mathbf{j}|} = \frac{a_2}{|\mathbf{a}|}, \quad \text{and} \tag{12}$$

$$\cos\gamma = \frac{\mathbf{a}\cdot\mathbf{k}}{|\mathbf{a}|\,|\mathbf{k}|} = \frac{a_3}{|\mathbf{a}|}.$$

That is, the direction cosines of **a** are the components of the *unit vector* $\mathbf{a}/|\mathbf{a}|$ with the same direction as **a**. Consequently

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \tag{13}$$

EXAMPLE 9 Find the direction angles of the vector $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

Solution Because $|\mathbf{a}| = \sqrt{14}$, the equations in (12) give

$$\alpha = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) \approx 57.69^\circ, \qquad \beta = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) \approx 36.70^\circ,$$

$$\text{and} \qquad \gamma = \cos^{-1}\left(\frac{-1}{\sqrt{14}}\right) \approx 105.50^\circ.$$

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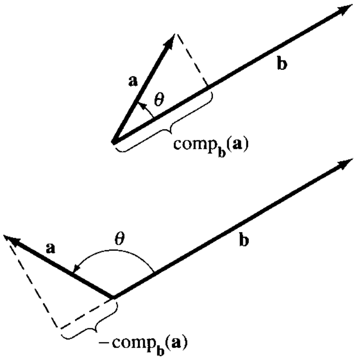

FIGURE 12.2.15 The component of **a** along **b**.

Sometimes we need to find the component of one vector **a** in the direction of another *nonzero* vector **b**. Think of the two vectors located with the same initial point (Fig. 12.2.15). Then the (scalar) **component of a along b**, denoted by $\operatorname{comp}_{\mathbf{b}}\mathbf{a}$, is numerically the length of the perpendicular projection of **a** onto the straight line determined by **b**. The number $\operatorname{comp}_{\mathbf{b}}\mathbf{a}$ is positive if the angle θ between **a** is acute (so **a** and **b** point in the same general direction) and negative if $\theta > \pi/2$. Thus $\operatorname{comp}_{\mathbf{b}}\mathbf{a} = |\mathbf{a}|\cos\theta$ in either case. Equation (10) then gives

$$\operatorname{comp}_{\mathbf{b}}\mathbf{a} = \frac{|\mathbf{a}|\,|\mathbf{b}|\cos\theta}{|\mathbf{b}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}. \tag{14}$$

There is no need to memorize this formula, for—in practice—we can always read $\operatorname{comp}_{\mathbf{b}}\mathbf{a} = |\mathbf{a}|\cos\theta$ from the figure and then apply Eq. (10) to eliminate $\cos\theta$. Note that $\operatorname{comp}_{\mathbf{b}}\mathbf{a}$ is a scalar, not a vector.

EXAMPLE 10 Given $\mathbf{a} = \langle 4, -5, 3\rangle$ and $\mathbf{b} = \langle 2, 1, -2\rangle$, express **a** as the sum of a vector $\mathbf{a}_{\parallel}$ parallel to **b** and a vector $\mathbf{a}_{\perp}$ perpendicular to **b**.

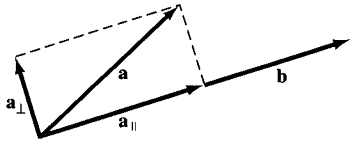

FIGURE 12.2.16 Construction of $\mathbf{a}_{\parallel}$ and $\mathbf{a}_{\perp}$.

Solution Our method of solution is motivated by the diagram in Fig. 12.2.16. We take

$$\mathbf{a}_{\parallel} = (\operatorname{comp}_{\mathbf{b}}\mathbf{a})\frac{\mathbf{b}}{|\mathbf{b}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} = \frac{8-5-6}{9}\mathbf{b}$$

$$= -\frac{1}{3}\langle 2, 1, -2\rangle = \left\langle -\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right\rangle,$$

and

$$\mathbf{a}_{\perp} = \mathbf{a} - \mathbf{a}_{\parallel} = \langle 4, -5, 3\rangle - \left\langle -\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right\rangle = \left\langle \frac{14}{3}, -\frac{14}{3}, \frac{7}{3}\right\rangle.$$

The diagram makes our choice of $\mathbf{a}_{\parallel}$ plausible, and we have deliberately chosen $\mathbf{a}_{\perp}$ so that $\mathbf{a} = \mathbf{a}_{\parallel} + \mathbf{a}_{\perp}$. To verify that the vector $\mathbf{a}_{\parallel}$ is indeed parallel to **b**, we simply note that it is a scalar multiple of **b**. To verify that $\mathbf{a}_{\perp}$ is perpendicular to **b**, we compute the dot product

$$\mathbf{a}_{\perp}\cdot\mathbf{b} = \tfrac{28}{3} - \tfrac{14}{3} - \tfrac{14}{3} = 0.$$

Thus $\mathbf{a}_{\parallel}$ and $\mathbf{a}_{\perp}$ have the required properties. ◆

One important application of vector components is to the definition and computation of *work*. Recall that the work W done by a constant force F exerted along the line of motion in moving a particle a distance d is given by $W = Fd$. But what if the force is a constant vector **F** pointing in some direction other than the line of motion, as when a child pulls a sled against the resistance of friction (Fig. 12.2.17)? Suppose that **F** moves a particle along the line segment from P to Q, and let $\mathbf{D} = \overrightarrow{PQ}$ be the resulting *displacement vector* of the object (Fig. 12.2.18). Then the **work** W done by the force **F** in moving the object along the line from P to Q is, by definition, the product of the component of **F** along **D** and the distance moved:

$$W = (\operatorname{comp}_{\mathbf{D}}\mathbf{F})\,|\mathbf{D}|. \tag{15}$$

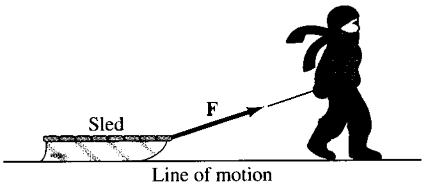

FIGURE 12.2.17 The vector force **F** is constant but acts at an angle to the line of motion (Example 10).

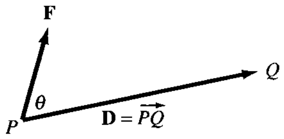

FIGURE 12.2.18 The force vector **F** and displacement vector **D** in Eq. (16).

If we use Eq. (14) and substitute $\text{comp}_{\mathbf{D}}\mathbf{F} = (\mathbf{F}\cdot\mathbf{D})/|\mathbf{D}|$, we get

$$W = \mathbf{F}\cdot\mathbf{D} \tag{16}$$

for the work done by the constant force **F** in moving an object along the displacement vector $\mathbf{D} = \overrightarrow{PQ}$. This formula is the vector generalization of the scalar work formula $W = Fd$. Work is measured in foot-pounds (ft·lb) if distance is measured in feet and force in pounds. If metric units of meters (m) for distance and newtons (N) for force are used, then work is measured in joules (J). (One joule is approximately 0.7376 ft·lb.)

EXAMPLE 11 Suppose that the force vector in Fig. 12.2.17 is inclined at an angle of 30° from the ground. If the child exerts a constant force of 20 lb, how much work is done in pulling the sled a distance of one mile?

Solution We are given that $|\mathbf{F}| = 20$ (lb) and $|\mathbf{D}| = 5280$ (ft). Because $\cos 30^\circ = \frac{1}{2}\sqrt{3}$, Eq. (16) yields

$$W = \mathbf{F}\cdot\mathbf{D} = |\mathbf{F}|\,|\mathbf{D}|\cos 30^\circ = (20)(5280)\left(\tfrac{1}{2}\sqrt{3}\right) \approx 91452 \quad \text{(ft·lb)}.$$

This may seem like a lot of work for a child to do. If the 1-mile trip takes an hour, then the child is generating *power* (work per unit time) at the rate of (91452 ft·lb)/(3600 s) ≈ 25.4 ft·lb/s. Because 1 horsepower (hp) is defined to be 550 ft·lb/s, the child's "power rating" is 25.4/550 ≈ $\frac{1}{20}$ hp. By comparison, an adult in excellent physical condition can climb the 2570 steps of the staircase of the CN tower in Toronto in less than 40 minutes. On October 29, 1989, Brendon Keenory of Toronto set the world's record for the fastest stairclimb there with a time of 7 min 52 s. Assuming that he climbed 1672 ft and weighed 160 lb, he generated an average of more than 0.988 hp over this time interval.

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12.2 TRUE/FALSE STUDY GUIDE

12.2 CONCEPTS: QUESTIONS AND DISCUSSION

1. Discuss the relation between a 3-dimensional vector and a point in space.
2. How does the dot product of two vectors resemble the ordinary product of two numbers? How do the two products differ?
3. Discuss the analogy between the absolute value of a number and the length of a vector.
4. Give an example of a real-world situation described by a triple of real numbers. In your example, do vector addition and scalar multiplication make any sense?

12.2 PROBLEMS

In Problems 1 through 6, find (a) $2\mathbf{a}+\mathbf{b}$, *(b)* $3\mathbf{a}-4\mathbf{b}$, *(c)* $\mathbf{a}\cdot\mathbf{b}$, *(d)* $|\mathbf{a}-\mathbf{b}|$, *and (e)* $\mathbf{a}/|\mathbf{a}|$.

1. $\mathbf{a} = \langle 2, 5, -4\rangle, \quad \mathbf{b} = \langle 1, -2, -3\rangle$

2. $\mathbf{a} = \langle -1, 0, 2\rangle, \quad \mathbf{b} = \langle 3, 4, -5\rangle$

3. $\mathbf{a} = \mathbf{i}+\mathbf{j}+\mathbf{k}, \quad \mathbf{b} = \mathbf{j}-\mathbf{k}$

4. $\mathbf{a} = 2\mathbf{i}-3\mathbf{j}+5\mathbf{k}, \quad \mathbf{b} = 5\mathbf{i}+3\mathbf{j}-7\mathbf{k}$

5. $\mathbf{a} = 2\mathbf{i}-\mathbf{j}, \quad \mathbf{b} = \mathbf{j}-3\mathbf{k}$

6. $\mathbf{a} = \mathbf{i}-2\mathbf{j}+3\mathbf{k}, \quad \mathbf{b} = \mathbf{i}+3\mathbf{j}-2\mathbf{k}$

7 through **12.** Find, to the nearest degree, the angle between the vectors **a** and **b** in Problems 1 through 6.

13 through **18.** Find $\text{comp}_{\mathbf{a}}\mathbf{b}$ and $\text{comp}_{\mathbf{b}}\mathbf{a}$ for the vectors **a** and **b** given in Problems 1 through 6.

In Problems 19 through 24, write the equation of the indicated sphere.

19. Center (3, 1, 2), radius 5

20. Center (−2, 1, −5), radius $\sqrt{7}$

21. One diameter: the segment joining (3, 5, −3) and (7, 3, 1)

22. Center (4, 5, −2), passing through the point (1, 0, 0)

23. Center (0, 0, 2), tangent to the xy-plane

24. Center (3, −4, 3), tangent to the xz-plane

In Problems 25 through 28, find the center and radius of the sphere with the given equation.

25. $x^2 + y^2 + z^2 + 4x - 6y = 0$

26. $x^2 + y^2 + z^2 - 8x - 9y + 10z + 40 = 0$

27. $3x^2 + 3y^2 + 3z^2 - 18z - 48 = 0$

28. $2x^2 + 2y^2 + 2z^2 = 7x + 9y + 11z$

In Problems 29 through 38, describe the graph of the given equation in geometric terms, using plain, clear language.

29. $z = 0$ **30.** $x = 0$

31. $z = 10$ **32.** $xy = 0$

33. $xyz = 0$ **34.** $x^2 + y^2 + z^2 + 7 = 0$

35. $x^2 + y^2 + z^2 = 0$ **36.** $x^2 + y^2 + z^2 - 2x + 1 = 0$

37. $x^2 + y^2 + z^2 - 6x + 8y + 25 = 0$

38. $x^2 + y^2 = 0$

Two vectors are ***parallel*** *provided that one is a scalar multiple of the other. Determine whether the vectors* **a** *and* **b** *in Problems 39 through 42 are parallel or perpendicular or neither.*

39. $\mathbf{a} = \langle 4, -2, 6\rangle$ and $\mathbf{b} = \langle 6, -3, 9\rangle$

40. $\mathbf{a} = \langle 4, -2, 6\rangle$ and $\mathbf{b} = \langle 4, 2, 2\rangle$

41. $\mathbf{a} = 12\mathbf{i} - 20\mathbf{j} + 16\mathbf{k}$ and $\mathbf{b} = -9\mathbf{i} + 15\mathbf{j} - 12\mathbf{k}$

42. $\mathbf{a} = 12\mathbf{i} - 20\mathbf{j} + 17\mathbf{k}$ and $\mathbf{b} = -9\mathbf{i} + 15\mathbf{j} + 24\mathbf{k}$

In Problems 43 and 44, determine whether or not the three given points lie on a single straight line.

43. $P(0, -2, 4)$, $Q(1, -3, 5)$, $R(4, -6, 8)$

44. $P(6, 7, 8)$, $Q(3, 3, 3)$, $R(12, 15, 18)$

In Problems 45 through 48, find (to the nearest degree) the three angles of the triangle with the given vertices.

45. $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$

46. $A(1, 0, 0)$, $B(1, 2, 0)$, $C(1, 2, 3)$

47. $A(1, 1, 1)$, $B(3, -2, 3)$, $C(3, 4, 6)$

48. $A(1, 0, 0)$, $B(0, 1, 0)$, $C(-1, -2, -2)$

In Problems 49 through 52, find the direction angles of the vector represented by $\overrightarrow{PQ}$.

49. $P(1, -1, 0)$, $Q(3, 4, 5)$

50. $P(2, -3, 5)$, $Q(1, 0, -1)$

51. $P(-1, -2, -3)$, $Q(5, 6, 7)$

52. $P(0, 0, 0)$, $Q(5, 12, 13)$

In Problems 53 and 54, find the work W done by the force **F** *in moving a particle in a straight line from P to Q.*

53. $\mathbf{F} = \mathbf{i} - \mathbf{k}$; $P(0, 0, 0)$, $Q(3, 1, 0)$

54. $\mathbf{F} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$; $P(5, 3, -4)$, $Q(-1, -2, 5)$

55. Suppose that the force vector in Fig. 12.2.17 is inclined at an angle of 40° from the ground. If the child exerts a constant force of 40 N, how much heat energy (in calories) does the child expend in pulling the sled a distance of 1 km along the ground? [*Note:* 1 J of work requires an expenditure of 0.239 calories of energy.]

56. A 1000-lb dog sled has a coefficient of sliding friction of 0.2, so it requires a force with a horizontal component of 200 lb to keep it moving at a constant speed. Suppose that a dog-team harness is attached so that the team's force vector makes an angle of 5° with the horizontal. If the dog team pulls this sled at a speed of 10 mi/h, how much power (in horsepower) are the dogs generating? [*Note:* 1 hp is 550 ft·lb/s.]

57. Suppose that the horizontal and vertical components of the three vectors shown in Fig. 12.2.19 balance (the algebraic sum of the horizontal components is zero, as is the sum of the vertical components). How much work is done by the constant force **F** (parallel to the inclined plane) in pulling the weight mg up the inclined plane a vertical height h?

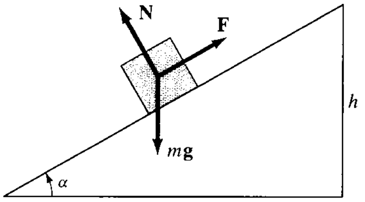


FIGURE 12.2.19 The inclined plane of Problem 57.

58. Prove the **Cauchy-Schwarz inequality:**

$$|\mathbf{a} \cdot \mathbf{b}| \leqq |\mathbf{a}|\,|\mathbf{b}|$$

for all pairs of vectors **a** and **b**.

59. Given two arbitrary vectors **a** and **b**, prove that they satisfy the **triangle inequality,**

$$|\mathbf{a} + \mathbf{b}| \leqq |\mathbf{a}| + |\mathbf{b}|.$$

[*Suggestion:* Square both sides.]

60. Prove that if **a** and **b** are arbitrary vectors, then

$$|\mathbf{a} - \mathbf{b}| \geqq |\mathbf{a}| - |\mathbf{b}|.$$

[*Suggestion:* Write $\mathbf{a} = (\mathbf{a} - \mathbf{b}) + \mathbf{b}$; then apply the triangle inequality of Problem 59.]

61. Use the dot product to construct a nonzero vector $\mathbf{w} = \langle w_1, w_2, w_3\rangle$ perpendicular to both of the vectors $\mathbf{u} = \langle 1, 2, -3\rangle$ and $\mathbf{v} = \langle 2, 0, 1\rangle$.

62. The unit cube in the first octant in space has opposite vertices $O(0, 0, 0)$ and $P(1, 1, 1)$. Find the angle between the edge of the cube on the x-axis and the diagonal OP.

63. Prove that the point M given in Eq. (2) is indeed the midpoint of the segment P_1P_2. [*Note:* You must prove *both* that M is equally distant from P_1 and P_2 *and* that M lies on the segment P_1P_2.]

64. Given vectors **a** and **b**, let $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$. Prove that the vector

$$\mathbf{c} = \frac{(b\mathbf{a} + a\mathbf{b})}{(a + b)}$$

bisects the angle between **a** and **b**.

65. Let **a**, **b**, and **c** be three vectors in the xy-plane with **a** and **b** nonzero and nonparallel. Show that there exist scalars α and β such that $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}$. [*Suggestion:* Begin by expressing **a**, **b**, and **c** in terms of **i**, **j**, and **k**.]

66. Let $ax + by + c = 0$ be the equation of the line L in the xy-plane with normal vector $\mathbf{n}$. Let $P_0(x_0, y_0)$ be a point on this line and $P_1(x_1, y_1)$ be a point not on L. Prove that the perpendicular distance from P_1 to L is

$$d = \frac{|\mathbf{n} \cdot \overrightarrow{P_0P_1}|}{|\mathbf{n}|} = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

67. Given the two points $A(3, -2, 4)$ and $B(5, 7, -1)$, write an equation in x, y, and z that says that the point $P(x, y, z)$ is equally distant from the points A and B. Then simplify this equation and give a geometric description of the set of all such points $P(x, y, z)$.

68. Given the fixed point $A(1, 3, 5)$, the point $P(x, y, z)$, and the vector $\mathbf{n} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, use the dot product to help you write an equation in x, y, and z that says this: $\mathbf{n}$ and $\overrightarrow{AP}$ are perpendicular. Then simplify this equation and give a geometric description of all such points $P(x, y, z)$.

69. Prove that the points $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$ are the vertices of a regular tetrahedron by showing that each of the six edges has length $\sqrt{2}$. Then use the dot product to find the angle between any two edges of the tetrahedron.

70. The methane molecule CH_4 is arranged with the four hydrogen atoms at the vertices of a regular tetrahedron and with the carbon atom at its center (Fig. 12.2.20). Suppose that the axes and scale are chosen so that the tetrahedron is that of Problem 69, with its center at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Find the *bond angle* α between the lines from the carbon atom to two of the hydrogen atoms.

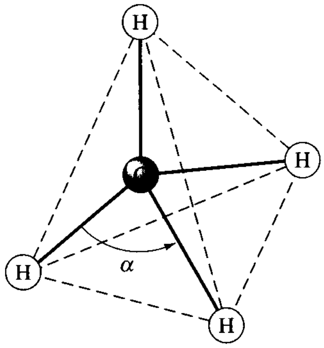


FIGURE 12.2.20 The methane bond angle α of Problem 70.

12.3 THE CROSS PRODUCT OF VECTORS

We often need to find a vector that is perpendicular to each of two vectors $\mathbf{a}$ and $\mathbf{b}$ in space. A routine way of doing this is provided by the *cross product* $\mathbf{a} \times \mathbf{b}$ of the vectors $\mathbf{a}$ and $\mathbf{b}$. This vector product is quite unlike the dot product $\mathbf{a} \cdot \mathbf{b}$ in that $\mathbf{a} \cdot \mathbf{b}$ is a *scalar*, whereas $\mathbf{a} \times \mathbf{b}$ is a *vector*. For this reason $\mathbf{a} \times \mathbf{b}$ is sometimes called the *vector product* of the two vectors $\mathbf{a}$ and $\mathbf{b}$.

The **cross product** (or **vector product**) of the vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is defined algebraically by the formula

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle. \tag{1}$$

Although this formula seems unmotivated, it has a redeeming feature: The product $\mathbf{a} \times \mathbf{b}$ is perpendicular both to $\mathbf{a}$ and to $\mathbf{b}$, as suggested in Fig. 12.3.1.

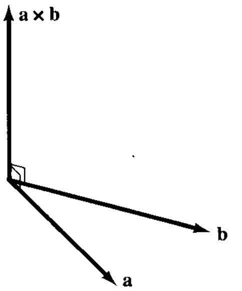


FIGURE 12.3.1 The cross product $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

THEOREM 1 Perpendicularity of the Cross Product

The cross product $\mathbf{a} \times \mathbf{b}$ is perpendicular both to $\mathbf{a}$ and to $\mathbf{b}$.

PROOF We show that $\mathbf{a} \times \mathbf{b}$ is perpendicular to $\mathbf{a}$ by showing that the dot product of $\mathbf{a}$ and $\mathbf{a} \times \mathbf{b}$ is zero. With the components as in Eq. (1), we find that

$$\begin{aligned}\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) &= a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1) \\ &= a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_2a_1b_3 + a_3a_1b_2 - a_3a_2b_1 \\ &= 0.\end{aligned}$$

A similar computation shows that $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ as well, so $\mathbf{a} \times \mathbf{b}$ is also perpendicular to the vector $\mathbf{b}$. ◀

You need not memorize Eq. (1), because there is an alternative version involving determinants that is easy both to remember and to use. Recall that a *determinant* of order 2 is defined as follows:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1. \tag{2}$$

EXAMPLE 1

$$\begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} = 2 \cdot 4 - (-1) \cdot 3 = 11.$$

◆

A determinant of order 3 can be defined in terms of determinants of order 2:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = +a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \tag{3}$$

Each element a_i of the first row is multiplied by the 2-by-2 "subdeterminant" obtained by deleting the row *and* column that contain a_i. Note in Eq. (3) that signs are attached to the a_i in accord with the checkerboard pattern

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}.$$

Equation (3) is an expansion of the 3-by-3 determinant along its first row. It can be expanded along any other row or column as well. For example, its expansion along its second column is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}.$$

In linear algebra it is shown that all such expansions yield the same value for the determinant.

Although we can expand a determinant of order 3 along any row or column, here we will use only expansions along the first row, as in Eq. (3) and Example 2.

EXAMPLE 2

$$\begin{aligned}\begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ -3 & 7 & 5 \end{vmatrix} &= 1 \cdot \begin{vmatrix} -1 & 4 \\ 7 & 5 \end{vmatrix} - 3 \cdot \begin{vmatrix} 2 & 4 \\ -3 & 5 \end{vmatrix} + (-2) \cdot \begin{vmatrix} 2 & -1 \\ -3 & 7 \end{vmatrix} \\ &= 1 \cdot (-5 - 28) + (-3) \cdot (10 + 12) + (-2) \cdot (14 - 3) \\ &= -33 - 66 - 22 = -121.\end{aligned}$$

◆

Equation (1) for the cross product of the vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is equivalent to

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}. \tag{4}$$

This is easy to verify by expanding the 2-by-2 determinants on the right-hand side and noting that the three components of the right-hand side of Eq. (1) result. Motivated by Eq. (4), we write

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \tag{5}$$

The "symbolic determinant" in this equation is to be evaluated by expansion along its first row, just as in Eq. (3) and just as though it were an ordinary determinant with real number entries. The result of this expansion is the right-hand side of Eq. (4). The components of the *first* vector **a** in $\mathbf{a} \times \mathbf{b}$ form the *second* row of the 3-by-3 determinant, and the components of the *second* vector **b** form the *third* row. The order of the vectors **a** and **b** is important because, as we soon shall see, $\mathbf{a} \times \mathbf{b}$ is generally *not* equal to $\mathbf{b} \times \mathbf{a}$: The cross product is *not commutative.*

Equation (5) for the cross product is the form most convenient for computational purposes.

EXAMPLE 3 If $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 2 & 2 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -1 \\ 2 & 2 \end{vmatrix}\mathbf{k}$$

$$= (1-4)\mathbf{i} - (-3-4)\mathbf{j} + (6-(-2))\mathbf{k}.$$

Thus

$$\mathbf{a} \times \mathbf{b} = -3\mathbf{i} + 7\mathbf{j} + 8\mathbf{k}.$$

You might now pause to verify (by using the dot product) that the vector $-3\mathbf{i} + 7\mathbf{j} + 8\mathbf{k}$ is perpendicular both to **a** and to **b**. ◆

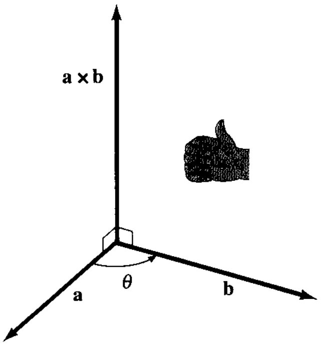


FIGURE 12.3.2 The vectors **a**, **b**, and $\mathbf{a} \times \mathbf{b}$—in that order—form a right-handed triple.

If the vectors **a** and **b** share the same initial point, then Theorem 1 implies that $\mathbf{a} \times \mathbf{b}$ is perpendicular to the plane determined by **a** and **b** (Fig. 12.3.2). There are still two possible directions for $\mathbf{a} \times \mathbf{b}$, but if $\mathbf{a} \times \mathbf{b} \neq \mathbf{0}$, then the triple $\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}$ is a *right-handed* triple in exactly the same sense as the triple **i**, **j**, **k**. Thus if the thumb of your right hand points in the direction of $\mathbf{a} \times \mathbf{b}$, then your fingers curl in the direction of rotation (less than 180°) from **a** to **b**.

Once we have established the direction of $\mathbf{a} \times \mathbf{b}$, we can describe the cross product in completely geometric terms by telling what the length $|\mathbf{a} \times \mathbf{b}|$ of the vector $\mathbf{a} \times \mathbf{b}$ is. This is given by the formula

$$|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2. \tag{6}$$

We can verify this vector identity routinely (though tediously) by writing $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, computing both sides of Eq. (6), and then noting that the results are equal (Problem 36).

Geometric Significance of the Cross Product

Equation (6) tells us what $|\mathbf{a} \times \mathbf{b}|$ is, but Theorem 2 reveals the geometric significance of the cross product.

THEOREM 2 Length of the Cross Product

Let θ be the angle between the nonzero vectors **a** and **b** (measured so that $0 \leqq \theta \leqq \pi$). Then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta. \tag{7}$$

PROOF We begin with Eq. (6) and use the fact that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$. Thus

$$|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (|\mathbf{a}|\,|\mathbf{b}|\cos\theta)^2$$
$$= |\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2\theta) = |\mathbf{a}|^2|\mathbf{b}|^2 \sin^2\theta.$$

Equation (7) now follows after we take the positive square root of both sides. (This is the correct root on the right-hand side because $\sin\theta \geqq 0$ for $0 \leqq \theta \leqq \pi$.) ◄

COROLLARY Parallel Vectors

Two nonzero vectors **a** and **b** are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

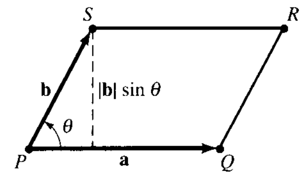


FIGURE 12.3.3 The area of the parallelogram $PQRS$ is $|\mathbf{a} \times \mathbf{b}|$.

In particular, the cross product of any vector with itself is the zero vector. Also, Eq. (1) shows immediately that the cross product of any vector with the zero vector is the zero vector itself. Thus

$$\mathbf{a} \times \mathbf{a} = \mathbf{a} \times \mathbf{0} = \mathbf{0} \times \mathbf{a} = \mathbf{0} \tag{8}$$

for every vector **a**.

Equation (7) has an important geometric interpretation. Suppose that **a** and **b** are represented by adjacent sides of a parallelogram $PQRS$, with $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{PS}$ (Fig. 12.3.3). The parallelogram then has base of length $|\mathbf{a}|$ and height $|\mathbf{b}| \sin\theta$, so its area is

$$A = |\mathbf{a}|\,|\mathbf{b}| \sin\theta = |\mathbf{a} \times \mathbf{b}|. \tag{9}$$

Thus *the length of the cross product* $\mathbf{a} \times \mathbf{b}$ *is numerically the same as the area of the parallelogram determined by* **a** *and* **b**. It follows that the area of the triangle PQS in Fig. 12.3.4, whose area is half that of the parallelogram, is

$$\tfrac{1}{2}A = \tfrac{1}{2}|\mathbf{a} \times \mathbf{b}| = \tfrac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PS}|. \tag{10}$$

Equation (10) gives a quick way to compute the area of a triangle—even one in space—without the need of finding any of its angles.

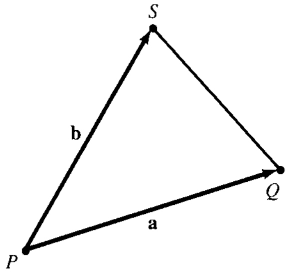


FIGURE 12.3.4 The area of ΔPQS is $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$.

EXAMPLE 4 Find the area of the triangle with vertices $A(3, 0, -1)$, $B(4, 2, 5)$, and $C(7, -2, 4)$.

Solution $\overrightarrow{AB} = \langle 1, 2, 6\rangle$ and $\overrightarrow{AC} = \langle 4, -2, 5\rangle$, so

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 6 \\ 4 & -2 & 5 \end{vmatrix} = 22\mathbf{i} + 19\mathbf{j} - 10\mathbf{k}.$$

Therefore, by Eq. (10), the area of triangle ABC is

$$\tfrac{1}{2}\sqrt{22^2 + 19^2 + (-10)^2} = \tfrac{1}{2}\sqrt{945} \approx 15.37.$$

◆

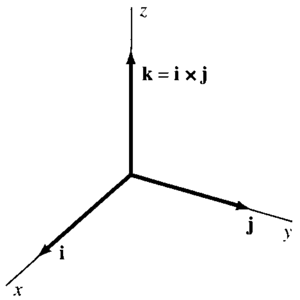


FIGURE 12.3.5 The basic unit vectors in space.

Now let **u**, **v**, **w** be a right-handed triple of mutually perpendicular *unit* vectors. The angle between any two of these is $\theta = \pi/2$, and $|\mathbf{u}| = |\mathbf{v}| = |\mathbf{w}| = 1$. Thus it follows from Eq. (7) that $\mathbf{u} \times \mathbf{v} = \mathbf{w}$. When we apply this observation to the basic unit vectors **i**, **j**, and **k** (Fig. 12.3.5), we see that

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \text{and} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}. \tag{11a}$$

But

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \text{and} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}. \tag{11b}$$

These observations, together with the fact that

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}, \tag{11c}$$

also follow directly from the original definition of the cross product [in the form in Eq. (5)]. The products in Eq. (11a) are easily remembered in terms of the sequence

$$\mathbf{i}, \ \mathbf{j}, \ \mathbf{k}, \ \mathbf{i}, \ \mathbf{j}, \ \mathbf{k}, \ \ldots.$$

The product of any two consecutive unit vectors, in the order in which they appear in this sequence, is the next one in the sequence.

Note *The cross product is not commutative:* $\mathbf{i} \times \mathbf{j} \neq \mathbf{j} \times \mathbf{i}$. Instead, it is **anticommutative:** For any two vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$. This is the first part of Theorem 3.

THEOREM 3 Algebraic Properties of the Cross Product

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and k is a real number, then

1. $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$; **(12)**
2. $(k\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (k\mathbf{b}) = k(\mathbf{a} \times \mathbf{b})$; **(13)**
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$; **(14)**
4. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$; **(15)**
5. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$. **(16)**

The proofs of Eqs. (12) through (15) are straightforward applications of the definition of the cross product in terms of components. See Problem 33 for an outline of the proof of Eq. (16).

We can find cross products of vectors expressed in terms of the basic unit vectors **i**, **j**, and **k** by means of computations that closely resemble those of ordinary algebra. We simply apply the algebraic properties summarized in Theorem 3 together with the relations in Eq. (11) giving the various products of the three unit vectors. We must be careful to preserve the order of factors, because vector multiplication is not commutative—although, of course, we should not hesitate to use Eq. (12).

EXAMPLE 5 $(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \times (3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$

$$\begin{aligned} &= 3(\mathbf{i} \times \mathbf{i}) + 2(\mathbf{i} \times \mathbf{j}) - 4(\mathbf{i} \times \mathbf{k}) - 6(\mathbf{j} \times \mathbf{i}) - 4(\mathbf{j} \times \mathbf{j}) \\ &\quad + 8(\mathbf{j} \times \mathbf{k}) + 9(\mathbf{k} \times \mathbf{i}) + 6(\mathbf{k} \times \mathbf{j}) - 12(\mathbf{k} \times \mathbf{k}) \\ &= 3 \cdot \mathbf{0} + 2\mathbf{k} - 4 \cdot (-\mathbf{j}) - 6 \cdot (-\mathbf{k}) - 4 \cdot \mathbf{0} + 8\mathbf{i} + 9\mathbf{j} + 6 \cdot (-\mathbf{i}) - 12 \cdot \mathbf{0} \\ &= 2\mathbf{i} + 13\mathbf{j} + 8\mathbf{k}. \end{aligned}$$

◆

Scalar Triple Products

Let us examine the product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ that appears in Eq. (15). This expression would not make sense were the parentheses instead around $\mathbf{a} \cdot \mathbf{b}$, because $\mathbf{a} \cdot \mathbf{b}$ is a scalar, and thus we could not form the cross product of $\mathbf{a} \cdot \mathbf{b}$ with the vector $\mathbf{c}$. This means that we may omit the parentheses—the expression $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ is not ambiguous—but we keep them for extra clarity. The dot product of the vectors $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$ is a real number, called the **scalar triple product** of the vectors **a**, **b**, and **c**. Equation (15) implies the curious fact that we can interchange the operations · (dot) and × (cross) without affecting the value of the expression:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

for all vectors **a**, **b**, and **c**.

To compute the scalar triple product in terms of components, write $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$. Then

$$\mathbf{b} \times \mathbf{c} = (b_2c_3 - b_3c_2)\mathbf{i} - (b_1c_3 - b_3c_1)\mathbf{j} + (b_1c_2 - b_2c_1)\mathbf{k},$$

so

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1).$$

But the expression on the right is the value of the 3-by-3 determinant

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \qquad \mathbf{(17)}$$

This is the quickest way to compute the scalar triple product.

EXAMPLE 6 If $\mathbf{a} = 2\mathbf{i} - 3\mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and $\mathbf{c} = 4\mathbf{j} - \mathbf{k}$, then

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & 0 & -3 \\ 1 & 1 & 1 \\ 0 & 4 & -1 \end{vmatrix}$$

$$= +2 \cdot \begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} + (-3) \cdot \begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix}$$

$$= 2 \cdot (-5) + (-3) \cdot 4 = -22.$$

◆

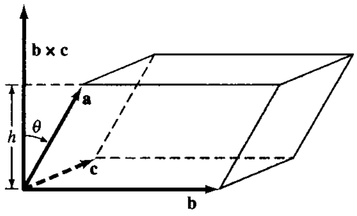


FIGURE 12.3.6 The volume of the parallelepiped is $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

The importance of the scalar triple product for applications depends on the following geometric interpretation. Let **a**, **b**, and **c** be three vectors with the same initial point. Figure 12.3.6 shows the parallelepiped determined by these vectors—that is, with arrows representing these vectors as adjacent edges. If the vectors **a**, **b**, and **c** are coplanar (lie in a single plane), then the parallelepiped is *degenerate* and its volume is zero. Theorem 4 holds whether or not the three vectors are coplanar, but it is most useful when they are not.

THEOREM 4 Scalar Triple Products and Volume

The volume V of the parallelepiped determined by the vectors **a**, **b**, and **c** is the absolute value of the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$; that is,

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|. \tag{18}$$

PROOF If the three vectors are coplanar, then **a** and $\mathbf{b} \times \mathbf{c}$ are perpendicular, so $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = 0$. Assume, then, that they are not coplanar. By Eq. (9) the area of the base (determined by **b** and **c**) of the parallelepiped is $A = |\mathbf{b} \times \mathbf{c}|$.

Now let α be the *acute* angle between **a** and the vector $\mathbf{b} \times \mathbf{c}$ that is perpendicular to the base. Then the height of the parallelepiped is $h = |\mathbf{a}| \cos \alpha$. If θ is the angle between the vectors **a** and $\mathbf{b} \times \mathbf{c}$, then either $\theta = \alpha$ or $\theta = \pi - \alpha$. Hence $\cos \alpha = |\cos \theta|$, so

$$V = Ah = |\mathbf{b} \times \mathbf{c}|\,|\mathbf{a}| \cos \alpha = |\mathbf{a}|\,|\mathbf{b} \times \mathbf{c}|\,|\cos \theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.$$

Thus we have verified Eq. (18). ◄

EXAMPLE 7 Figure 12.3.7 shows the pyramid $OPQR$ and the parallelepiped both determined by the vectors

$$\mathbf{a} = \overrightarrow{OP} = \langle 3, 2, -1 \rangle, \quad \mathbf{b} = \overrightarrow{OQ} = \langle -2, 5, 1 \rangle, \quad \text{and} \quad \mathbf{c} = \overrightarrow{OR} = \langle 2, 1, 5 \rangle.$$

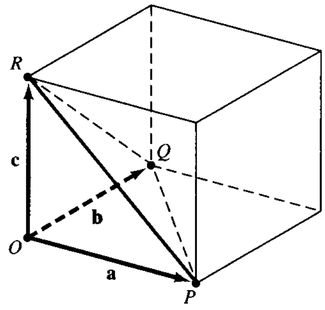


FIGURE 12.3.7 The pyramid (and parallelepiped) of Example 7.

The volume of the pyramid is $V = \frac{1}{3}Ah$, where h is its height and the area A of its base OPQ is *half* the area of the corresponding base of the parallelepiped. It therefore follows from Eq. (17) and (18) that V is one-sixth the volume of the parallelepiped:

$$V = \frac{1}{6}|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = \frac{1}{6}\begin{vmatrix} 3 & 2 & -1 \\ -2 & 5 & 1 \\ 2 & 1 & 5 \end{vmatrix} = \frac{108}{6} = 18.$$

◆

EXAMPLE 8 Use the scalar triple product to show that the points $A(1, -1, 2)$, $B(2, 0, 1)$, $C(3, 2, 0)$, and $D(5, 4, -2)$ are coplanar.

Solution It's enough to show that the vectors $\overrightarrow{AB} = \langle 1, 1, -1 \rangle$, $\overrightarrow{AC} = \langle 2, 3, -2 \rangle$, and $\overrightarrow{AD} = \langle 4, 5, -4 \rangle$ are coplanar. But their scalar triple product is

$$\begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & -2 \\ 4 & 5 & -4 \end{vmatrix} = 1 \cdot (-2) - 1 \cdot 0 + (-1) \cdot (-2) = 0,$$

so Theorem 4 guarantees that the parallelepiped determined by these three vectors has volume zero. Hence the four given points are coplanar. ◆

The cross product occurs quite often in scientific applications. For example, suppose that a body in space is free to rotate around the fixed point O. If a force $\mathbf{F}$ acts at a point P of the body, that force causes the body to rotate. This effect is measured by the **torque vector** τ defined by the relation

$$\tau = \mathbf{r} \times \mathbf{F},$$

where $\mathbf{r} = \overrightarrow{OP}$, the straight line through O determined by τ is the axis of rotation, and the length

$$|\tau| = |\mathbf{r}|\,|\mathbf{F}|\sin\theta$$

is the **moment** of the force $\mathbf{F}$ around this axis (Fig. 12.3.8).

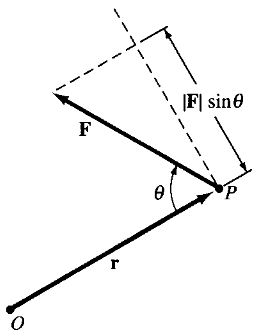


FIGURE 12.3.8 The torque vector τ is normal to both $\mathbf{r}$ and $\mathbf{F}$.

Another application of the cross product involves the force exerted on a moving charged particle by a magnetic field. This force is important in particle accelerators, mass spectrometers, and television picture tubes; controlling the paths of the ions is accomplished through the interplay of electric and magnetic fields. In such circumstances, the force $\mathbf{F}$ on the particle due to a magnetic field depends on three things: the charge q of the particle, its velocity vector $\mathbf{v}$, and the magnetic field vector $\mathbf{B}$ at the instantaneous location of the particle. And it turns out that

$$\mathbf{F} = (q\mathbf{v}) \times \mathbf{B}.$$

12.3 TRUE/FALSE STUDY GUIDE

12.3 CONCEPTS: QUESTIONS AND DISCUSSION

1. How does the cross product of two vectors resemble the ordinary product of two numbers? How do the two products differ?
2. Discuss the differences and the similarities between the dot product and the cross product of two vectors.
3. A surveyor measures a polygonal plot of land by first finding the coordinates of the vertices of its bounding polygon. Outline how the surveyor might then proceed to use cross products to calculate the area of the plot.

12.3 PROBLEMS

Find $\mathbf{a} \times \mathbf{b}$ *in Problems 1 through 4.*

1. $\mathbf{a} = \langle 5, -1, -2\rangle, \quad \mathbf{b} = \langle -3, 2, 4\rangle$
2. $\mathbf{a} = \langle 3, -2, 0\rangle, \quad \mathbf{b} = \langle 0, 3, -2\rangle$
3. $\mathbf{a} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}, \quad \mathbf{b} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$
4. $\mathbf{a} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}, \quad \mathbf{b} = 2\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$

In Problems 5 and 6, find the cross product of the given 2-dimensional vectors $\mathbf{a} = \langle a_1, a_2\rangle$ *and* $\mathbf{b} = \langle b_1, b_2\rangle$ *by first "extending" them to 3-dimensional vectors* $\mathbf{a} = \langle a_1, a_2, 0\rangle$ *and* $\mathbf{b} = \langle b_1, b_2, 0\rangle$.

5. $\mathbf{a} = \langle 2, -3\rangle$ and $\mathbf{b} = \langle 4, 5\rangle$
6. $\mathbf{a} = -5\mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = 7\mathbf{i} - 11\mathbf{j}$

In Problems 7 and 8, find two different unit vectors $\mathbf{u}$ *and* $\mathbf{v}$ *both of which are perpendicular to both the given vectors* $\mathbf{a}$ *and* $\mathbf{b}$.

7. $\mathbf{a} = \langle 3, 12, 0\rangle$ and $\mathbf{b} = \langle 0, 4, 3\rangle$
8. $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$
9. Apply Eq. (5) to verify the equations in (11a).
10. Apply Eq. (5) to verify the equations in (11b).
11. Prove that the vector product is not associative by comparing $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ with $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ in the case $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{i} + \mathbf{j}$, and $\mathbf{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.
12. Find nonzero vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ such that $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ but $\mathbf{b} \neq \mathbf{c}$.
13. Suppose that the three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are mutually perpendicular. Prove that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{0}$.
14. Find the area of the triangle with vertices $P(1, 1, 0)$, $Q(1, 0, 1)$, and $R(0, 1, 1)$.
15. Find the area of the triangle with vertices $P(1, 3, -2)$, $Q(2, 4, 5)$, and $R(-3, -2, 2)$.
16. Find the volume of the parallelepiped with adjacent edges $\overrightarrow{OP}$, $\overrightarrow{OQ}$, and $\overrightarrow{OR}$, where P, Q, and R are the points given in Problem 14.

17. (a) Find the volume of the parallelepiped with adjacent edges $\overrightarrow{OP}$, $\overrightarrow{OQ}$, and $\overrightarrow{OR}$, where P, Q, and R are the points given in Problem 15. (b) Find the volume of the pyramid with vertices O, P, Q, and R.

18. Find a unit vector $\mathbf{n}$ perpendicular to the plane through the points P, Q, and R of Problem 15. Then find the distance from the origin to this plane by computing $\mathbf{n} \cdot \overrightarrow{OP}$.

In Problems 19 through 22, determine whether or not the four given points A, B, C, and D are coplanar. If not, find the volume of the pyramid with these four points as its vertices, given that its volume is one-sixth that of the parallelepiped spanned by $\overrightarrow{AB}$, $\overrightarrow{AC}$, *and* $\overrightarrow{AD}$.

19. $A(1, 3, -2)$, $B(3, 4, 1)$, $C(2, 0, -2)$, and $D(4, 8, 4)$

20. $A(13, -25, -37)$, $B(25, -14, -22)$, $C(24, -38, -25)$, and $D(26, 10, -19)$

21. $A(5, 2, -3)$, $B(6, 4, 0)$, $C(7, 5, 1)$, and $D(14, 14, 18)$

22. $A(25, 22, -33)$, $B(36, 34, -20)$, $C(27, 25, -29)$, and $D(34, 34, -12)$

23. Figure 12.3.9 shows a polygonal plot of land, with angles and lengths measured by a surveyor. First find the coordinates of each vertex. Then use the vector product [as in Eq. (10)] to calculate the area of the plot.

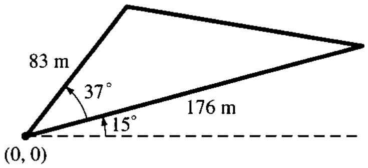


FIGURE 12.3.9 Problem 23.

24. Repeat Problem 23 with the plot shown in Fig. 12.3.10.

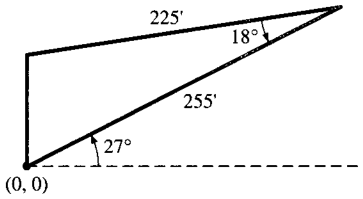


FIGURE 12.3.10 Problem 24.

25. Repeat Problem 23 with the plot shown in Fig. 12.3.11. [*Suggestion:* First divide the plot into two triangles.]

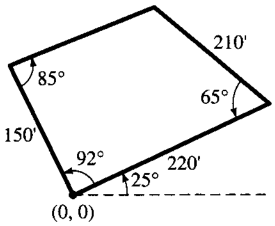


FIGURE 12.3.11 Problem 25.

26. Repeat Problem 23 with the plot shown in Fig. 12.3.12.

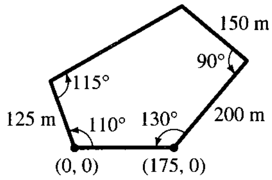


FIGURE 12.3.12 Problem 26.

27. Apply Eq. (5) to verify Eq. (12), the anticommutativity of the vector product.

28. Apply Eq. (17) to verify the identity for scalar triple products stated in Eq. (15).

29. Suppose that P and Q are points on a line L in space. Let A be a point not on L (Fig. 12.3.13). (a) Calculate in two ways the area of the triangle APQ to show that the perpendicular distance from A to the line L is

$$d = \frac{|\overrightarrow{AP} \times \overrightarrow{AQ}|}{|\overrightarrow{PQ}|}.$$

(b) Use this formula to compute the distance from the point $(1, 0, 1)$ to the line through the two points $P(2, 3, 1)$ and $Q(-3, 1, 4)$.

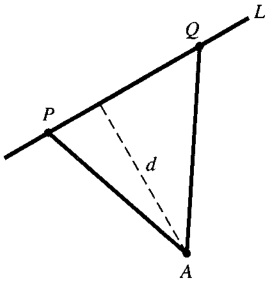


FIGURE 12.3.13 Problem 29.

30. Suppose that A is a point not on the plane determined by the three points P, Q, and R. Calculate in two ways the volume of the pyramid $APQR$ to show that the perpendicular distance from A to this plane is

$$d = \frac{|\overrightarrow{AP} \cdot (\overrightarrow{AQ} \times \overrightarrow{AR})|}{|\overrightarrow{PQ} \times \overrightarrow{PR}|}.$$

Use this formula to compute the distance from the point $(1, 0, 1)$ to the plane through the points $P(2, 3, 1)$, $Q(3, -1, 4)$, and $R(0, 0, 2)$.

31. Suppose that P_1 and Q_1 are two points on the line L_1 and that P_2 and Q_2 are two points on the line L_2. If the lines L_1 and L_2 are not parallel, then the perpendicular distance d between them is the projection of $\overrightarrow{P_1P_2}$ onto a vector $\mathbf{n}$ that is perpendicular both to $\overrightarrow{P_1Q_1}$ and $\overrightarrow{P_2Q_2}$. Prove that

$$d = \frac{|\overrightarrow{P_1P_2} \cdot (\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2})|}{|\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}|}.$$

32. Use the following method to establish that the **vector triple product** $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ is equal to $(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$. (a) Let $\mathbf{I}$ be a unit vector in the direction of $\mathbf{a}$ and let $\mathbf{J}$ be a unit vector perpendicular to $\mathbf{I}$ and parallel to the plane of $\mathbf{a}$ and $\mathbf{b}$. Let $\mathbf{K} = \mathbf{I} \times \mathbf{J}$. Explain why there are scalars a_1, b_1, b_2, c_1, c_2, and c_3 such that

$$\mathbf{a} = a_1\mathbf{I}, \quad \mathbf{b} = b_1\mathbf{I} + b_2\mathbf{J}, \quad \text{and} \quad \mathbf{c} = c_1\mathbf{I} + c_2\mathbf{J} + c_3\mathbf{K}.$$

(b) Now show that

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -a_1b_2c_2\mathbf{I} + a_1b_2c_1\mathbf{J}.$$

(c) Finally, substitute for $\mathbf{I}$ and $\mathbf{J}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

33. By permutation of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, deduce from Problem 32 that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

[this is Eq. (16)].

34. Deduce from the orthogonality properties of the vector product that the vector $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$ can be written in the form $r_1\mathbf{a} + r_2\mathbf{b}$ and in the form $s_1\mathbf{c} + s_2\mathbf{d}$.

35. Consider the triangle in the xy-plane that has vertices $(x_1, y_1, 0)$, $(x_2, y_2, 0)$, and $(x_3, y_3, 0)$. Use the vector product to prove that the area of this triangle is *half* the *absolute value* of the determinant

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}.$$

36. Given the vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, verify Eq. (6),

$$|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2\,|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2,$$

by computing each side in terms of the components of $\mathbf{a}$ and $\mathbf{b}$.

12.4 LINES AND PLANES IN SPACE

Just as in the plane, a straight line in space is determined by any two points P_0 and P_1 that lie on it. We may write $\mathbf{v} = \overrightarrow{P_0P_1}$—meaning that the directed line segment $\overrightarrow{P_0P_1}$ represents the vector $\mathbf{v}$—to describe the "direction of the line." Thus, alternatively, a line in space can be specified by giving a point P_0 on it *and* a [nonzero] vector $\mathbf{v}$ that determines the direction of the line.

To investigate equations that describe lines in space, let us begin with a straight line L that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ (Fig. 12.4.1). Then another point $P(x, y, z)$ lies on the line L if and only if the vectors $\mathbf{v}$ and $\overrightarrow{P_0P}$ are parallel, in which case

$$\overrightarrow{P_0P} = t\mathbf{v} \tag{1}$$

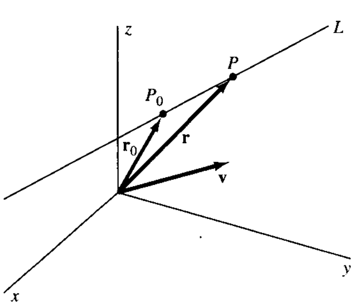


FIGURE 12.4.1 Finding the equation of the line L that passes through the point P_0 and is parallel to the vector $\mathbf{v}$.

for some real number t. If $\mathbf{r}_0 = \overrightarrow{OP_0}$ and $\mathbf{r} = \overrightarrow{OP}$ are the position vectors of the points P_0 and P, respectively, then $\overrightarrow{P_0P} = \mathbf{r} - \mathbf{r}_0$. Hence Eq. (1) gives the **vector equation**

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} \tag{2}$$

describing the line L. As indicated in Fig. 12.4.1, $\mathbf{r}$ is the position vector of an *arbitrary* point P on the line L, and Eq. (2) gives $\mathbf{r}$ in terms of the parameter t, the position vector $\mathbf{r}_0$ of a *fixed* point P_0 on L, and the fixed vector $\mathbf{v}$ that determines the direction of L.

The left- and right-hand sides of Eq. (2) are equal, and each side is a vector. So corresponding components are also equal. When we write the resulting equations, we get a scalar description of the line L. Because $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ and $\mathbf{r} = \langle x, y, z \rangle$, Eq. (2) thereby yields the three scalar equations

$$x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct. \tag{3}$$

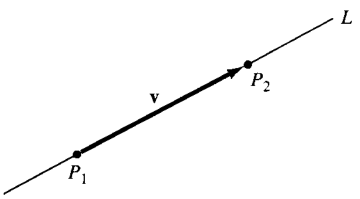


FIGURE 12.4.2 The line L of Example 1.

These are **parametric equations** of the line L that passes through the point (x_0, y_0, z_0) and is parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$.

EXAMPLE 1 Write parametric equations of the line L that passes through the points $P_1(1, 2, 2)$ and $P_2(3, -1, 3)$ of Fig. 12.4.2.

Solution The line L is parallel to the vector

$$\mathbf{v} = \overrightarrow{P_1P_2} = (3\mathbf{i} - \mathbf{j} + 3\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k},$$

so we take $a = 2$, $b = -3$, and $c = 1$. With P_1 as the fixed point, the equations in (3) give

$$x = 1 + 2t, \qquad y = 2 - 3t, \qquad z = 2 + t$$

as parametric equations of L. In contrast, with P_2 as the fixed point and with the vector

$$-2\mathbf{v} = -4\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$$

(parallel to $\mathbf{v}$) as the direction vector, the equations in (3) yield the parametric equations

$$x = 3 - 4t, \qquad y = -1 + 6t, \qquad z = 3 - 2t.$$

Thus the parametric equations of a line are not unique. ◆

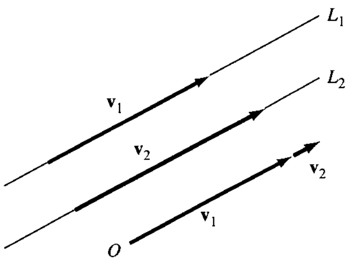


FIGURE 12.4.3 Parallel lines.

Given two straight lines L_1 and L_2 with parametric equations

$$x = x_1 + a_1 t, \qquad y = y_1 + b_1 t, \qquad z = z_1 + c_1 t \tag{4}$$

and

$$x = x_2 + a_2 s, \qquad y = y_2 + b_2 s, \qquad z = z_2 + c_2 s, \tag{5}$$

respectively, we can see at a glance whether or not L_1 and L_2 are parallel. Because L_1 is parallel to $\mathbf{v}_1 = \langle a_1, b_1, c_1 \rangle$ and L_2 is parallel to $\mathbf{v}_2 = \langle a_2, b_2, c_2 \rangle$, it follows that the lines L_1 and L_2 are parallel if and only if the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are scalar multiples of each other (Fig. 12.4.3). If the two lines are not parallel, we can attempt to find a point of intersection by solving the equations

$$x_1 + a_1 t = x_2 + a_2 s \quad \text{and} \quad y_1 + b_1 t = y_2 + b_2 s$$

simultaneously for s and t. If these values of s and t also satisfy the equation $z_1 + c_1 t = z_2 + c_2 s$, then we have found a point of intersection. Its rectangular coordinates can be found by substituting the resulting value of t into Eq. (4) [or the resulting value of s into Eq. (5)]. Otherwise, the lines L_1 and L_2 do not intersect. Two nonparallel and nonintersecting lines in space are called **skew lines** (Fig. 12.4.4).

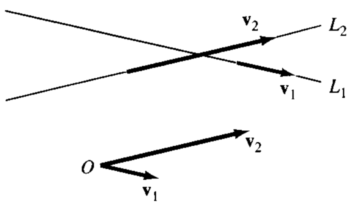


FIGURE 12.4.4 Skew lines.

EXAMPLE 2 The line L_1 with parametric equations

$$x = 1 + 2t, \qquad y = 2 - 3t, \qquad z = 2 + t$$

passes through the point $P_1(1, 2, 2)$ (discovered by substituting $t = 0$) and is parallel to the vector $\mathbf{v}_1 = \langle 2, -3, 1 \rangle$. The line L_2 with parametric equations

$$x = 3 + 4t, \qquad y = 1 - 6t, \qquad z = 5 + 2t$$

passes through the point $P_2(3, 1, 5)$ and is parallel to the vector $\mathbf{v}_2 = \langle 4, -6, 2 \rangle$. Because $\mathbf{v}_2 = 2\mathbf{v}_1$, we see that L_1 and L_2 are parallel.

But are L_1 and L_2 actually different lines, or are we perhaps dealing with two different parametrizations of the same line? To answer this question, we note that $\overrightarrow{P_1P_2} = \langle 2, -1, 3 \rangle$ is not a multiple of, and therefore is not parallel to, $\mathbf{v}_1 = \langle 2, -3, 1 \rangle$. Thus the point P_2 does not lie on the line L_1, and hence the lines L_1 and L_2 are indeed distinct. ◆

If all the coefficients a, b, and c in (3) are nonzero, then we can eliminate the parameter t. Simply solve each equation for t and then set the resulting expressions equal to each other. This gives

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}. \tag{6}$$

These are called the **symmetric equations** of the line L. If one or more of a or b or c is zero, this means that L lies in a plane parallel to one of the coordinate planes, and in this case the line does not have symmetric equations. For example, if $c = 0$, then L lies in the horizontal plane $z = z_0$. Of course, it is still possible to write equations for L that don't include the parameter t; if $c = 0$, for instance, but a and b are nonzero, then we could describe the line L as the set of points (x, y, z) satisfying the equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}, \qquad z = z_0.$$

EXAMPLE 3 Find both parametric and symmetric equations of the line L through the points $P_0(3, 1, -2)$ and $P_1(4, -1, 1)$. Find also the points at which L intersects the three coordinate planes.

Solution The line L is parallel to the vector $\mathbf{v} = \overrightarrow{P_0P_1} = \langle 1, -2, 3\rangle$, so we take $a = 1$, $b = -2$, and $c = 3$. The equations in (3) then give the parametric equations

$$x = 3 + t, \qquad y = 1 - 2t, \qquad z = -2 + 3t$$

of L, and the equations in (6) give the symmetric equations

$$\frac{x-3}{1} = \frac{y-1}{-2} = \frac{z+2}{3}.$$

To find the point at which L intersects the xy-plane, we set $z = 0$ in the symmetric equations. This gives

$$\frac{x-3}{1} = \frac{y-1}{-2} = \frac{2}{3},$$

and so $x = \frac{11}{3}$ and $y = -\frac{1}{3}$. Thus L meets the xy-plane at the point $(\frac{11}{3}, -\frac{1}{3}, 0)$. Similarly, $x = 0$ gives $(0, 7, -11)$ for the point where L meets the yz-plane, and $y = 0$ gives $(\frac{7}{2}, 0, -\frac{1}{2})$ for its intersection with the xz-plane. ◆

Planes in Space

A plane $\mathcal{P}$ in space is determined by a point $P_0(x_0, y_0, z_0)$ through which $\mathcal{P}$ passes and a line through P_0 that is normal to $\mathcal{P}$. Alternatively, we may be given P_0 on $\mathcal{P}$ and a vector $\mathbf{n} = \langle a, b, c\rangle$ normal to the plane $\mathcal{P}$. The point $P(x, y, z)$ lies on the plane $\mathcal{P}$ if and only if the vectors $\mathbf{n}$ and $\overrightarrow{P_0P}$ are perpendicular (Fig. 12.4.5), in which case $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$. We write $\overrightarrow{P_0P} = \mathbf{r} - \mathbf{r}_0$, where $\mathbf{r}$ and $\mathbf{r}_0$ are the position vectors $\mathbf{r} = \overrightarrow{OP}$ and $\mathbf{r}_0 = \overrightarrow{OP_0}$ of the points P and P_0, respectively. Thus we obtain a **vector equation**

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \tag{7}$$

of the plane $\mathcal{P}$.

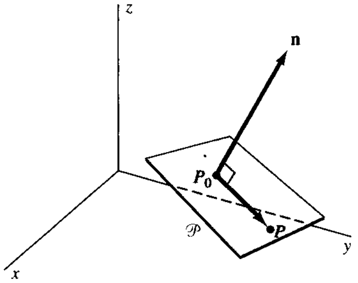


FIGURE 12.4.5 Because **n** is normal to $\mathcal{P}$, it follows that **n** is normal to $\overrightarrow{P_0P}$ for all points P in $\mathcal{P}$.

If we substitute $\mathbf{n} = \langle a, b, c\rangle$, $\mathbf{r} = \langle x, y, z\rangle$, and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ into Eq. (7), we thereby obtain a **scalar equation**

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{8}$$

of the plane through $P_0(x_0, y_0, z_0)$ with **normal vector** $\mathbf{n} = \langle a, b, c\rangle$.

EXAMPLE 4 An equation of the plane through $P_0(-1, 5, 2)$ with normal vector $\mathbf{n} = \langle 1, -3, 2\rangle$ is

$$1 \cdot (x + 1) + (-3) \cdot (y - 5) + 2 \cdot (z - 2) = 0;$$

that is, $x - 3y + 2z = -12$. ◆

IMPORTANT The coefficients of x, y, and z in the last equation are the components of the normal vector. This is always the case, because we can write Eq. (8) in the form

$$ax + by + cz = d, \tag{9}$$

where $d = ax_0 + by_0 + cz_0$. Conversely, every *linear equation* in x, y, and z of the form in Eq. (9) represents a plane in space provided that the coefficients a, b, and c are not all zero. The reason is that if $c \neq 0$ (for instance), then we can choose x_0 and y_0 arbitrarily and solve the equation $ax_0 + by_0 + cz_0 = d$ for z_0. With these values, Eq. (9) takes the form

$$ax + by + cz = ax_0 + by_0 + cz_0;$$

that is,

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

so this equation represents the plane through (x_0, y_0, z_0) with normal vector $\langle a, b, c\rangle$.

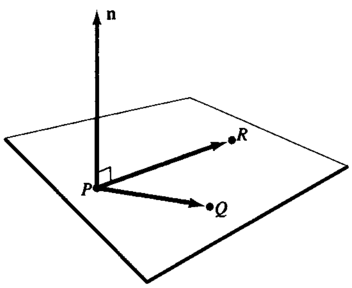


FIGURE 12.4.6 The normal vector **n** as a cross product (Example 5).

EXAMPLE 5 Find an equation for the plane through the three points $P(2, 4, -3)$, $Q(3, 7, -1)$, and $R(4, 3, 0)$.

Solution We want to use Eq. (8), so we first need a vector **n** that is normal to the plane in question. One easy way to obtain such a normal vector is by using the cross product. Let

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 2 \\ 2 & -1 & 3 \end{vmatrix} = 11\mathbf{i} + \mathbf{j} - 7\mathbf{k}.$$

Because $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are in the plane, their cross product **n** is normal to the plane (Fig. 12.4.6). Hence the plane has equation

$$11(x - 2) + (y - 4) - 7(z + 3) = 0.$$

After simplifying, we write the equation as

$$11x + y - 7z = 47.$$

◆

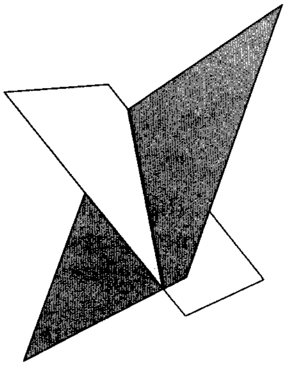


FIGURE 12.4.7 The intersection of two nonparallel planes is a straight line.

Two planes with normal vectors **n** and **m** are said to be **parallel** provided that **n** and **m** are parallel. Otherwise, the two planes meet in a straight line (Fig. 12.4.7), and we can find the angle θ between the normal vectors **n** and **m** (Fig. 12.4.8). We then define the **angle** between the two planes to be either θ or $\pi - \theta$, whichever is an acute angle.

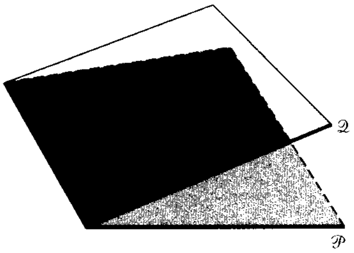


FIGURE 12.4.8 Vectors **m** and **n** normal to the planes $\mathcal{P}$ and $\mathcal{Q}$, respectively.

EXAMPLE 6 Find the angle θ between the planes with equations

$$2x + 3y - z = -3 \quad \text{and} \quad 4x + 5y + z = 1.$$

Then write symmetric equations of their line of intersection L.

Solution The vectors $\mathbf{n} = \langle 2, 3, -1\rangle$ and $\mathbf{m} = \langle 4, 5, 1\rangle$ are normal to the two planes, so

$$\cos\theta = \frac{\mathbf{n}\cdot\mathbf{m}}{|\mathbf{n}|\,|\mathbf{m}|} = \frac{22}{\sqrt{14}\sqrt{42}}.$$

Hence $\theta = \cos^{-1}(\frac{11}{21}\sqrt{3}) \approx 24.87°$.

To determine the line of intersection L of the two planes, we need first to find a point P_0 that lies on L. We can do this by substituting an arbitrarily chosen value of x into the equations of the given planes and then solving the resulting equations for y and z. With $x = 1$ we get the equations

$$2 + 3y - z = -3,$$
$$4 + 5y + z = 1.$$

The common solution is $y = -1$, $z = 2$. Thus the point $P_0(1, -1, 2)$ lies on the line L.

Next we need a vector **v** parallel to L. The vectors **n** and **m** normal to the two planes are both perpendicular to L, so their cross product is parallel to L. Alternatively, we can find a second point P_1 on L by substituting a second value of x into the equations of the given planes and solving for y and z, as before. With $x = 5$ we obtain the equations

$$\begin{aligned} 10 + 3y - z &= -3, \\ 20 + 5y + z &= 1, \end{aligned}$$

with common solution $y = -4$, $z = 1$. Thus we obtain a second point $P_1(5, -4, 1)$ on L and thereby the vector

$$\mathbf{v} = \overrightarrow{P_0P_1} = \langle 4, -3, -1 \rangle$$

parallel to L. From Eq. (6) we now find symmetric equations

$$\frac{x-1}{4} = \frac{y+1}{-3} = \frac{z-2}{-1}$$

of the line of intersection of the two given planes. ◆

Finally, we may note that the symmetric equations of a line L present the line as an intersection of planes: We can rewrite the equations in (6) in the form

$$\begin{aligned} b(x - x_0) - a(y - y_0) &= 0, \\ c(x - x_0) - a(z - z_0) &= 0, \\ c(y - y_0) - b(z - z_0) &= 0. \end{aligned} \qquad \textbf{(10)}$$

These are the equations of three planes that intersect in the line L. The first has normal vector $\langle b, -a, 0\rangle$, a vector parallel to the xy-plane. So the first plane is perpendicular to the xy-plane. Similarly, the second plane is perpendicular to the xz-plane and the third is perpendicular to the yz-plane.

The equations in (10) are symmetric equations of the line that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to $\mathbf{v} = \langle a, b, c\rangle$. Unlike the equations in (6), these equations are meaningful whether or not all the components a, b, and c are nonzero. They have a special form, though, if one of the three components is zero. If, say, $a = 0$, then the first two equations in (10) take the form $x = x_0$. The line is then the intersection of the two planes $x = x_0$ and $c(y - y_0) = b(z - z_0)$.

EXAMPLE 7 In Example 3 we saw that the line L through the point $P_0(3, 1, -2)$ and $P_1(4, -1, 1)$ has symmetric equations

$$\frac{x-3}{1} = \frac{y-1}{-2} = \frac{z+2}{3}.$$

Proceeding to rewrite these equations as in (10), we obtain first the equations

$$\begin{aligned} -2(x-3) &= \quad y - 1, \\ 3(x-3) &= \quad z + 2, \\ 3(y-1) &= -2(z+2) \end{aligned}$$

and then (upon simplification) the equations

$$\begin{aligned} 2x + \ y &= \ 7, \\ 3x - \ z &= 11, \\ 3y + 2z &= -1 \end{aligned}$$

that represent L as the intersection of three planes, each of them parallel to one of the three coordinate axes in space. Figure 12.4.9 shows a computer plot of these three planes intersecting in the line L. ◆

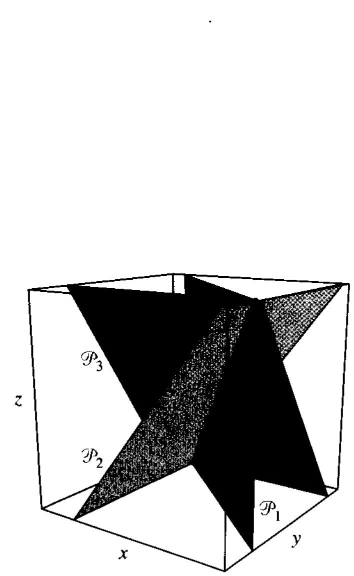
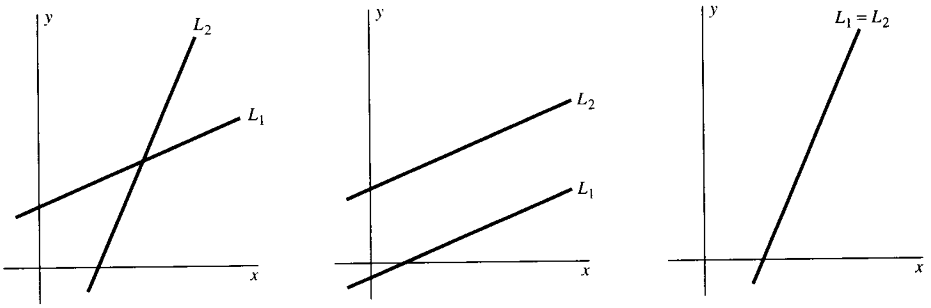


FIGURE 12.4.9 The line L of Example 7 is the intersection of the plane $\mathcal{P}_1$ parallel to the z-axis, the plane $\mathcal{P}_2$ parallel to the y-axis, and the plane $\mathcal{P}_3$ parallel to the x-axis.

12.4 TRUE/FALSE STUDY GUIDE

12.4 CONCEPTS: QUESTIONS AND DISCUSSION

1. Figure 12.4.10 shows the possible configuration of two lines L_1 and L_2 in the xy-plane. We see that the intersection of L_1 and L_2 can consists of either one point, no points, or infinitely many points. Explain why this geometric observation implies that two linear equations $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ in two unknowns x and y can have either a single simultaneous solution (x, y), no solution, or infinitely many different solutions.



(a) Two intersecting lines: a unique solution.

(b) Two parallel lines: no solution.

(c) Two coincident lines: infinitely many solutions.

FIGURE 12.4.10 (a) The nonparallel lines L_1 and L_2 intersect in a single point. (b) The distinct parallel lines L_1 and L_2 have no point of intersection. (c) The coincident lines L_1 and L_2 have infinitely many points in common.

In each of the following cases, describe similarly the possible configurations and hence the possible number of points of intersection of the indicated number of lines or planes. Translate your geometric conclusion into a statement about the possible number of solutions of a system of two or three linear equations in two or three unknowns.

2. Three lines in the plane.
3. Two lines in space.
3. Three planes in space.

12.4 PROBLEMS

In Problems 1 through 4, write parametric equations of the straight line that passes through the point P and is parallel to the vector **v**.

1. $P(0, 0, 0)$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
2. $P(3, -4, 5)$, $\mathbf{v} = -2\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}$
3. $P(4, 13, -3)$, $\mathbf{v} = 2\mathbf{i} - 3\mathbf{k}$
4. $P(17, -13, -31)$, $\mathbf{v} = \langle -17, 13, 31\rangle$

In Problems 5 through 8, write parametric equations of the straight line that passes through the points P_1 and P_2.

5. $P_1(0, 0, 0)$, $P_2(-6, 3, 5)$
6. $P_1(3, 5, 7)$, $P_2(6, -8, 10)$
7. $P_1(3, 5, 7)$, $P_2(6, 5, 4)$
8. $P_1(29, -47, 13)$, $P_2(73, 53, -67)$

In Problems 9 through 14, write both parametric and symmetric equations for the indicated straight line.

9. Through $P(2, 3, -4)$ and parallel to $\mathbf{v} = \langle 1, -1, -2\rangle$
10. Through $P(2, 5, -7)$ and $Q(4, 3, 8)$
11. Through $P(1, 1, 1)$ and perpendicular to the xy-plane
12. Through the origin and perpendicular to the plane with equation $x + y + z = 1$
13. Through $P(2, -3, 4)$ and perpendicular to the plane with equation $2x - y + 3z = 4$
14. Through $P(2, -1, 5)$ and parallel to the line with parametric equations $x = 3t$, $y = 2 + t$, $z = 2 - t$

In Problems 15 through 20, determine whether the two lines L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

15. L_1: $x - 2 = \frac{1}{2}(y + 1) = \frac{1}{3}(z - 3)$;
    L_2: $\frac{1}{3}(x - 5) = \frac{1}{2}(y - 1) = z - 4$
16. L_1: $\frac{1}{4}(x - 11) = y - 6 = -\frac{1}{2}(z + 5)$;
    L_2: $\frac{1}{6}(x - 13) = -\frac{1}{3}(y - 2) = \frac{1}{8}(z - 5)$
17. L_1: $x = 6 + 2t$, $y = 5 + 2t$, $z = 7 + 3t$;
    L_2: $x = 7 + 3s$, $y = 5 + 3s$, $z = 10 + 5s$
18. L_1: $x = 14 + 3t$, $y = 7 + 2t$, $z = 21 + 5t$;
    L_2: $x = 5 + 3s$, $y = 15 + 5s$, $z = 10 + 7s$

19. L_1: $\frac{1}{6}(x-7)=\frac{1}{4}(y+5)=-\frac{1}{8}(z-9)$;
L_2: $-\frac{1}{9}(x-11)=-\frac{1}{6}(y-7)=\frac{1}{12}(z-13)$

20. L_1: $x=13+12t, \quad y=-7+20t, \quad z=11-28t$;
L_2: $x=22+9s, \quad y=8+15s, \quad z=-10-21s$

In Problems 21 through 24, write an equation of the plane with normal vector **n** *that passes through the point P.*

21. $P(0,0,0), \quad \mathbf{n}=\langle 1,2,3\rangle$

22. $P(3,-4,5), \quad \mathbf{n}=\langle -2,7,3\rangle$

23. $P(5,12,13), \quad \mathbf{n}=\mathbf{i}-\mathbf{k}$

24. $P(5,12,13), \quad \mathbf{n}=\mathbf{j}$

In Problems 25 through 32, write an equation of the indicated plane.

25. Through $P(5,7,-6)$ and parallel to the xz-plane

26. Through $P(1,0,-1)$ with normal vector $\mathbf{n}=\langle 2,2,-1\rangle$

27. Through $P(10,4,-3)$ with normal vector $\mathbf{n}=\langle 7,11,0\rangle$

28. Through $P(1,-3,2)$ with normal vector $\mathbf{n}=\overrightarrow{OP}$

29. Through the origin and parallel to the plane with equation $3x+4y=z+10$

30. Through $P(5,1,4)$ and parallel to the plane with equation $x+y-2z=0$

31. Through the origin and the points $P(1,1,1)$ and $Q(1,-1,3)$

32. Through the points $A(1,0,-1)$, $B(3,3,2)$, and $C(4,5,-1)$

In Problems 33 and 34, write an equation of the plane that contains both the point P and the line L.

33. $P(2,4,6)$; $\quad L: x=7-3t, \quad y=3+4t, \quad z=5+2t$

34. $P(13,-7,29)$; $\quad L: x=17-9t, \quad y=23+14t, \quad z=35-41t$

In Problems 35 through 38, determine whether the line L and the plane $\mathcal{P}$ intersect or are parallel. If they intersect, find the point of intersection.

35. L: $x=7-4t, \quad y=3+6t, \quad z=9+5t$;
$\mathcal{P}$: $4x+y+2z=17$

36. L: $x=15+7t, \quad y=10+12t, \quad z=5-4t$;
$\mathcal{P}$: $12x-5y+6z=50$

37. L: $x=3+2t, \quad y=6-5t, \quad z=2+3t$;
$\mathcal{P}$: $3x+2y-4z=1$

38. L: $x=15-3t, \quad y=6-5t, \quad z=21-14t$;
$\mathcal{P}$: $23x+29y-31z=99$

In Problems 39 through 42, find the angle between the planes with the given equations.

39. $x=10$ and $x+y+z=0$

40. $2x-y+z=5$ and $x+y-z=1$

41. $x-y-2z=1$ and $x-y-2z=5$

42. $2x+y+z=4$ and $3x-y-z=3$

In Problems 43 through 46, write both parametric and symmetric equations of the line of intersection of the indicated planes.

43. The planes of Problem 39

44. The planes of Problem 40

45. The planes of Problem 41

46. The planes of Problem 42

47. Write symmetric equations for the line through $P(3,3,1)$ that is parallel to the line of Problem 46.

48. Find an equation of the plane through $P(3,3,1)$ that is perpendicular to the planes $x+y=2z$ and $2x+z=10$.

49. Find an equation of the plane through $(1,1,1)$ that intersects the xy-plane in the same line as does the plane $3x+2y-z=6$.

50. Find an equation for the plane that passes through the point $P(1,3,-2)$ and contains the line of intersection of the planes $x-y+z=1$ and $x+y-z=1$.

51. Find an equation of the plane that passes through the points $P(1,0,-1)$ and $Q(2,1,0)$ and is parallel to the line of intersection of the planes $x+y+z=5$ and $3x-y=4$.

52. Prove that the lines $x-1=\frac{1}{2}(y+1)=z-2$ and $x-2=\frac{1}{3}(y-2)=\frac{1}{2}(z-4)$ intersect. Find an equation of the [only] plane that contains them both.

53. Prove that the line of intersection of the planes $x+2y-z=2$ and $3x+2y+2z=7$ is parallel to the line $x=1+6t$, $y=3-5t$, $z=2-4t$. Find an equation of the plane determined by these two lines.

54. Show that the perpendicular distance D from the point $P_0(x_0, y_0, z_0)$ to the plane $ax+by+cz=d$ is

$$D=\frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}.$$

[*Suggestion:* The line that passes through P_0 and is perpendicular to the given plane has parametric equations $x=x_0+at$, $y=y_0+bt$, $z=z_0+ct$. Let $P_1(x_1, y_1, z_1)$ be the point of this line, corresponding to $t=t_1$, at which it intersects the given plane. Solve for t_1, and then compute $D=|\overrightarrow{P_0P_1}|$.]

In Problems 55 and 56, use the formula of Problem 54 to find the distance between the given point and the given plane.

55. The origin and the plane $x+y+z=10$

56. The point $P(5,12,-13)$ and the plane with equation $3x+4y+5z=12$

57. Prove that any two skew lines lie in parallel planes.

58. Use the formula of Problem 54 to show that the perpendicular distance D between the two parallel planes $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ is

$$D=\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}.$$

59. The line L_1 is described by the equations

$$x-1=2y+2, \qquad z=4.$$

The line L_2 passes through the points $P(2,1,-3)$ and $Q(0,8,4)$. (a) Show that L_1 and L_2 are skew lines. (b) Use the results of Problems 57 and 58 to find the perpendicular distance between L_1 and L_2.

60. Find the shortest distance between points of the line L_1 with parametric equations

$$x=7+2t, \quad y=11-5t, \quad z=13+6t$$

and the line L_2 of intersection of the planes $3x-2y+4z=10$ and $5x+3y-2z=15$.