1. For the points $A(2, 2, -3)$, $B(1, 1, -3)$ and $C(1, 2, -2)$ find.

a) Equation of the plane through these 3 points. (5 pts)

$\vec{AB} = \langle -1, -1, 0 \rangle$

$\vec{AC} = \langle -1, 0, 1 \rangle$

$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$

$= -\vec{i} + \vec{j} - \vec{k}$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$-1(x-2) + 1(y-2) - 1(z+3) = 0$

$-x + y - z = 3$ #

b) Area of the triangle with vertices at these points. (5 pts)

Area of $\Delta = \frac{1}{2} |\vec{AB} \times \vec{AC}|$

$= \frac{1}{2} \sqrt{(-1)^2 + 1^2 + (-1)^2}$

$= \frac{\sqrt{3}}{2}$ #

c) The angle between vectors $\vec{AB}$ and $\vec{BC}$. (5 pts)

$\vec{AB} = \langle -1, -1, 0 \rangle \qquad |\vec{AB}| = \sqrt{1 + 1 + 0} = \sqrt{2}$

$\vec{BC} = \langle 0, 1, 1 \rangle \qquad |\vec{BC}| = \sqrt{0 + 1 + 1} = \sqrt{2}$

$\cos\theta = \frac{\vec{AB} \cdot \vec{BC}}{|\vec{AB}||\vec{BC}|} = \frac{0 - 1 + 0}{\sqrt{2} \cdot \sqrt{2}}$

$\cos\theta = -\frac{1}{2}$

$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ #

2. (5 pts each)

a) Find spherical coordinates of a point with rectangular coordinates $(-1, 1, -\sqrt{6})$

ρ, ϕ, θ

$(x, y, z) = (-1, 1, -\sqrt{6})$

$\rho = \sqrt{x^2+y^2+z^2}$

$= \sqrt{1+1+6} = \sqrt{8}$

$\rho = 2\sqrt{2}$

$\phi = \cos^{-1}\left(\frac{z}{\rho}\right)$

$= \cos^{-1}\left(\frac{-\sqrt{6}}{2\sqrt{2}}\right)$

$= \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$\phi = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$\tan\theta = \frac{y}{x}$

$\tan\theta = \frac{1}{-1} = -1$ — 2nd Quadrant

$\theta = \pi - \frac{\pi}{4}$

$\theta = \frac{3\pi}{4}$

$\therefore$ spherical coord. $= \left(2\sqrt{2}, \frac{5\pi}{6}, \frac{3\pi}{4}\right)$

✓

b) The equation of a surface in spherical coordinates is

$$\rho \sin\varphi = 3/\sin\theta$$

Describe the surface. (**Hint:** the surface is a very simple one!)

$\rho \sin\phi = \frac{3}{\sin\theta}$

$\rho \sin\phi \sin\theta = 3$

$y = 3$

$\therefore$ The surface is the plane $y = 3$ #

✓

3. Evaluate the limit or show that it does not exist. (10 pts each)

a) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{\sin(x^2-y^2)}{x^2-y^2}$

let $t = x^2 - y^2$ as $(x,y) \to (1,1)$, $t \to 0$

$\therefore \displaystyle\lim_{(x,y)\to(1,1)} \frac{\sin(x^2-y^2)}{x^2-y^2} = \lim_{t\to 0} \frac{\sin t}{t}$

$\displaystyle = \lim_{t\to 0} \frac{\cos t}{1}$ ✓

$= 1$ #

b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{3x^2+2y^2}$

let $y = mx$

$\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{3x^2+2y^2} = \lim_{x\to 0} \frac{mx^2}{3x^2+2m^2x^2}$

$\displaystyle = \lim_{x\to 0} \frac{mx^2}{(3+2m^2)x^2}$

$\displaystyle = \frac{m}{3+2m^2}$ ✓

if $m=1$, the limit approaches $\frac{1}{3+2} = \frac{1}{5}$

if $m=-1$, the limit approaches $\frac{-1}{3+2} = -\frac{1}{5}$

$\therefore$ Because the limit approaches different values from different directions,

the limit of $\frac{xy}{3x^2+2y^2}$ as $(x,y) \to (0,0)$ does not exist.

4. (10 pts) The acceleration **a** of a particle is given by the formula

$$\mathbf{a} = \mathbf{i}\sin 2t - \mathbf{j}\cos 2t \qquad \bar{r}$$

Assume that at time $t = 0$ the particle is at the point $(0, 1/4)$ and has initial velocity $\mathbf{v}_0 = -1/2\mathbf{i}$. $\bar{v}$ $\frac{1}{4}\bar{j}$

Show that the path of the particle is a circle. What is its radius?

Recall that the points on a circle have the same distance from its center.

$$\bar{a} = \bar{i}\sin 2t - \bar{j}\cos 2t$$

$$\bar{v} = \int (\bar{i}\sin 2t - \bar{j}\cos 2t)\,dt$$

$$\bar{v}(t) = -\frac{1}{2}\cos 2t\,\bar{i} - \frac{1}{2}\sin 2t\,\bar{j} + C_1$$

$$\bar{v}(0) = -\frac{1}{2}\cos 0\,\bar{i} - \frac{1}{2}\sin 0\,\bar{j} + C_1$$

$$-\frac{1}{2}\bar{i} = -\frac{1}{2}\bar{i} - 0 + C_1$$

$$\therefore C_1 = 0$$

$$\therefore \bar{v}(t) = -\frac{1}{2}\cos 2t\,\bar{i} - \frac{1}{2}\sin 2t\,\bar{j}$$

$$\bar{r}(t) = \int \left(-\frac{1}{2}\cos 2t\,\bar{i} - \frac{1}{2}\sin 2t\,\bar{j}\right)dt$$

$$\bar{r}(t) = -\frac{1}{4}\sin 2t\,\bar{i} + \frac{1}{4}\cos 2t\,\bar{j} + C_2 \qquad \checkmark$$

$$\bar{r}(0) = -\frac{1}{4}\sin 0\,\bar{i} + \frac{1}{4}\cos 0\,\bar{j} + C_2$$

$$\frac{1}{4}\bar{j} = 0 + \frac{1}{4}\bar{j} + C_2$$

$$\therefore C_2 = 0$$

$$\therefore \bar{r}(t) = -\frac{1}{4}\sin 2t\,\bar{i} + \frac{1}{4}\cos 2t\,\bar{j}$$

$$= \left\langle -\frac{1}{4}\sin 2t, \frac{1}{4}\cos 2t \right\rangle = \langle x(t), y(t) \rangle$$

$$|\bar{r}(t)| = \sqrt{\left(-\frac{1}{4}\sin 2t\right)^2 + \left(\frac{1}{4}\cos 2t\right)^2} = \sqrt{\frac{1}{16}\sin^2 2t + \frac{1}{16}\cos^2 2t}$$

$$= \sqrt{\frac{1}{16}(\sin^2 2t + \cos^2 2t)}$$

$$= \sqrt{\frac{1}{16}}$$

$$|\bar{r}(t)| = \frac{1}{4}$$

$\therefore$ the path of the particle is a circle with radius of $\frac{1}{4}$ #

$x^2 + y^2 = \frac{1}{16}$

5. For the surface

$$z = e^{x/y^2}$$

a) (10 pts) Find the tangent plane to the surface at the point $(1, 1, 1)$.

$$z = e^{x/y^2}$$

$$\frac{\partial z}{\partial x} = \frac{1}{y^2} e^{x/y^2}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,1)} = \frac{1}{1} e^{1/1} = e$$

$$\frac{\partial z}{\partial y} = -\frac{2x}{y^3} e^{x/y^2}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,1)} = -2e$$

$$z - 1 = e(x-1) - 2e(y-1)$$

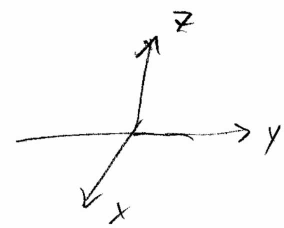
$$z - 1 = ex - e - 2ey + 2e$$

$$\therefore \quad ex - 2ey - z = -e - 1$$ — tangent plane

b) (10 pts) Compute the angle between this tangent plane and the xy-plane. Remember that the angle between two planes is between 0 and $\pi/2$.

Terms of the form "$\arccos(\sqrt{2}/3)$" are perfectly acceptable.

From (a); normal vector of the tangent plane is $\langle e, -2e, -1 \rangle = \bar{a}$



let the normal vector of the xy-plane be $\langle 0, 0, -1 \rangle = \bar{b}$

$$|\bar{a}| = \sqrt{e^2 + 4e^2 + 1} = \sqrt{5e^2+1}$$

$$|\bar{b}| = 1$$

$$\cos\theta = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}||\bar{b}|}$$

$$\cos\theta = \frac{0 + 0 + 1}{\sqrt{5e^2+1} \cdot 1} = \frac{1}{\sqrt{5e^2+1}}$$

$$\therefore \quad \theta = \arccos\left(\frac{1}{\sqrt{5e^2+1}}\right) \#$$

6. (10 pts) A particle moves on a sphere of radius 10 centered at 0 (i.e. the position vector $\mathbf{r}(t)$ satisfies $|\mathbf{r}(t)| = 10$ for all t). Differentiating $|\mathbf{r}|^2 = \mathbf{r} \cdot \mathbf{r}$ show that velocity is always orthogonal to the position vector.

$$\vec{r} \cdot \vec{r} = 10$$

$$(\vec{r} \cdot \vec{r})' = 0$$

$$\vec{r} \cdot \vec{r}\,' + \vec{r}\,' \cdot \vec{r} = 0$$

$$2\vec{r} \cdot \vec{r}\,' = 0$$

$$\vec{r} \cdot \vec{r}\,' = 0$$

So $\vec{r}$ is orthogonal to $\vec{r}\,' = \vec{v}$

7. (15 pts) Find a point (x, y) in the plane for which the sum of the squares of its distances from $(0,0)$, $(1,0)$ and $(0,2)$ is minimal.

dist. $(0,0)$ and (x,y) $= \sqrt{x^2+y^2}$

dist. $(1,0)$ and (x,y) $= \sqrt{(x-1)^2+y^2}$

dist. $(0,2)$ and (x,y) $= \sqrt{x^2+(y-2)^2}$

$$S(x,y) = x^2+y^2+(x-1)^2+y^2+x^2+(y-2)^2$$

$$= x^2+y^2+x^2-2x+1+y^2+x^2+y^2-4y+4$$

$$S(x,y) = 3x^2-2x+3y^2-4y+5$$

$$\frac{\partial S}{\partial x} = 6x-2 = 0$$

$$x = \frac{1}{3}$$

$$\frac{\partial S}{\partial y} = 6y-4 = 0$$

$$y = \frac{2}{3}$$

$\therefore$ a point $(\frac{1}{3}, \frac{2}{3})$ in the plane for which the sum is minimal.

Why min?

$S(x,y) \to \infty$ as $x^2+y^2 \to \infty$

so critical point is minimum.

(unique)