9/12 Michael Morris

$$C_p[a,b] = \{ f : f \text{ continuous}, \left(\int |f(x)|^p dx\right)^{1/p} < \infty \}$$

This is not complete.

<u>Def</u>: $\underline{L^p(a,b)}$ is the completion of $C_p[a,b]$

We could also have $R_p[a,b]$ which would be Riemann intergrable but would only have a semi-norm and not be complete.

<u>Alternate Def</u>: $\underline{L^p}$ is the space of measurable functions s.t. $\int_a^b |f(x)|^p dx < \infty$ using Lebesgue Integration.

Let $1 \leq p \leq p_1 \leq \infty$. To show: $L^{p_1}(a,b) \subset L^p(a,b)$

$f \in L^{p_1}$ Let $r = \frac{p_1}{p} \geq 1$ $\frac{1}{r} + \frac{1}{r'} = 1$

$$\int_a^b |f(x)|^p \cdot 1 \, dx \leq \left(\int_a^b |f|^{pr}\right)^{1/r} \left(\int_a^b 1^{r'}\right)^{1/r'}$$

$$\leq \left(\int_a^b |f|^{p_1}\right)^{1/r} (b-a)^{1/r'} < \infty \quad \Rightarrow f \in L^p \quad \blacksquare$$

<u>Bounded Linear Operators (Ch. 4)</u>

<u>Def</u> Let X, Y be vector spaces (or linear spaces), then T is <u>linear</u> if $T(\alpha \vec{x}_1 + \beta \vec{x}_2) = \alpha T\vec{x}_1 + \beta T\vec{x}_2 \quad \forall \vec{x}_1, \vec{x}_2 \in X \; \forall \alpha, \beta \in \mathbb{F}$

<u>Def</u> $T: X \to Y$ linear, X, Y are normed. Then T is <u>bounded</u> if $\exists \; C < \infty$ s.t. $\|T\vec{x}\| \leq C\|\vec{x}\| \quad \forall x \in X$.

<u>Def</u>: $\inf \{ C : \|Tx\| \leq C\|x\| \; \forall x \in X \} =: \|T\|$, the <u>norm of T</u>.

<u>Recall</u>: infimum and supremum

Let $A \subset \mathbb{R}$

A bdd above, i.e. $\exists \; M$ s.t. $x \leq M \; \forall x \in A$

sup A is the least upper bound. The number $\bar{a}$ s.t. $x \leq \bar{a} \; \forall x \in A$ but $\forall \varepsilon > 0 \; \exists x \in A$ s.t. $x > \bar{a} - \varepsilon$

Let A bdd below, i.e. $\exists\ m$ s.t. $x \geq m \ \forall x \in A$.
$\inf A$ is the greatest lower bound. The number $\underline{a}$ s.t. $x \geq \underline{a} \ \forall x \in A$
but $\forall \varepsilon > 0 \ \exists x \in A$ s.t. $x < \underline{a} + \varepsilon$.

Clearly $\|Tx\| \leq \|T\|\|x\|$ by the definition of $\|T\|$.

<u>Thm</u>: $\|T\| = \sup \{ \|Tx\| : x \in \mathcal{X}, \ \|x\| \leq 1 \}$ (1)

$= \quad " \quad\quad " \quad \|x\| = 1 \}$ (2)

$= \quad " \quad\quad " \quad \|x\| < 1 \}$ (3)

$= \sup \left\{ \dfrac{\|Tx\|}{\|x\|} : x \in \mathcal{X}, \ x \neq \vec{0} \right\}$ (4)

<u>Pf</u>: $\|Tx\| \leq \|T\|\|x\|$ ($\leq$) If $\|x\| = 1 \Rightarrow \|Tx\| \leq \|T\|$

$\Rightarrow \sup \{ \|Tx\| : \|x\| = 1 \} \leq \|T\|$

($\geq$) $\forall \varepsilon > 0 \ \exists x$ s.t. $\|Tx\| > (\|T\| - \varepsilon)\|x\|$

$\Rightarrow \left\| T \frac{x}{\|x\|} \right\| \geq \|T\| - \varepsilon$

$\sup \{ \|Tx\| ; \|x\| = 1 \} > \|T\| - \varepsilon$

$\Rightarrow \sup \{ \|Tx\| : \|x\| = 1 \} \geq \|T\|$

This complete the proof of (2) and (4).

<u>Note</u>: $\|T\|$ is a norm on $\mathcal{L}(\mathcal{X} \to Y)$.

<u>Def</u>: <u>$\mathcal{L}(\mathcal{X} \to Y)$</u> is the space of bounded linear operators from $\mathcal{X}$ to Y.

It is trivial to show $\|T\|$ is a norm except for the triangle ineq.

$\|A + B\| = \sup \{ \|Ax + Bx\| : \|x\| = 1 \}$

$\leq \sup \{ \|Ax\| + \|Bx\| : \|x\| = 1 \}$

$\leq \sup \{ \|A\| + \|B\| : \|x\| = 1 \} = \|A\| + \|B\|$.

Def : $B(\mathfrak{X}, Y) = \mathcal{L}(\mathfrak{X}, Y)$.

Thm : $\mathfrak{X}$ normed, Y Banach, then $B(\mathfrak{X}, Y)$ is Banach.

Pf : Textbook

Def: $\mathfrak{X}_0 \subset \mathfrak{X}$ is *dense* if $\forall x \in \mathfrak{X}\ \exists \{x_n\}_1^\infty$ $x_n \in \mathfrak{X}_0$ s.t. $\|x - x_n\| \to 0$.

Extension of a bdd operator from a dense set.

$\mathfrak{X}$ is normed, Y is Banach.

$\mathfrak{X}_0 \subset \mathfrak{X}$, $\mathfrak{X}_0$ is dense and linear (subspace)

Let $T_0 \in B(\mathfrak{X}_0, Y)$

Thm : $\exists!\ T \in B(\mathfrak{X}, Y)$ s.t. $Tx_0 = T_0 x_0$ $\forall x_0 \in \mathfrak{X}_0$

$\mathfrak{X}, \ell^p$ $\quad p < \infty$ $\quad \mathfrak{X}_0$ - "finite" sequences in ℓ^p