Lemma: If $T \in B(X,Y)$. $x_n \to x$ $\left(\lim_{n\to\infty} \|x_n - x\| = 0\right)$.

Then $Tx_n \to Tx$ $\left(\|Tx_n - Tx\| \xrightarrow[n\to\infty]{} 0\right)$.

Pf $0 \le \|Tx_n - Tx\| \le \|T\| \cdot \|x_n - x\|$

$\downarrow$

$\to 0 \Rightarrow$ whole thing converges to 0.

Pinching Principle $\Rightarrow 0 \to 0 \Rightarrow \lim_{n\to\infty} \|Tx_n - Tx\| = 0$.

This lemma means:

If T is an extension of T_0 and $x_n \in X_0$ and $x_n \to x$,

then Tx must be $\lim Tx_n$ (must show limit exists).

$T_0 x_n$.

Pf Take $x \in X$ and $x_n \in X_0$ s.t. $x_n \to x$.

$\{x_n\}_1^\infty$ is Cauchy, so $\exists N\ \forall \varepsilon > 0$ s.t. $\forall n,m > N$ $\|x_n - x_m\| < \frac{\varepsilon}{\|T_0\|}$.

Then $\|T_0 x_n - T_0 x_m\| < \|T_0\| \frac{\varepsilon}{\|T_0\|} = \varepsilon$.

So $\{T_0 x_n\}_1^\infty$ is Cauchy.

$\Rightarrow \exists \lim_{n\to\infty} T_0 x_n \equiv Tx$. (Still need to show that if $x_n \to x$, $\tilde{x}_n \to x$ then $\lim T_0 x_n = \lim T_0 \tilde{x}$

But $\|T_0 x_n - T_0 \tilde{x}_n\| \le \|T_0\| \|x_n - \tilde{x}_n\|$

$\to 0$.

Then use Pinching.

$\Rightarrow$ Well defined.

Need to show linear:

$T(\alpha x_1 + \beta x_2) = \alpha Tx_1 + \beta Tx_2$

(trivial using limit properties).

Need to show T is bounded:

If $x_n \to x$ then $\|x_n\| \to \|x\|$ (Exercise).

$\|T_0 x_n\| \le \|T_0\| \|x_n\|$

$\|Tx\| \le \|T_0\| \|x\|$. So $\|T\| \le \|T_0\|$.

Remark. $\|T\| = \|T_0\|$ because $\|T_0\| \le \|T\|$ is trivial.

$\Rightarrow$ $\|T\| = \sup\{\|Tx\| : x \in X, \|x\| \le 1\}$

$\|T_0\| = \sup\{\|Tx\| : x \in X_0, \|x\| \le 1\}$ $\Leftarrow$ you take sup of smaller set $\Rightarrow$ must be smaller

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$\varepsilon/3$ Theorem and Uniform Boundedness Principle

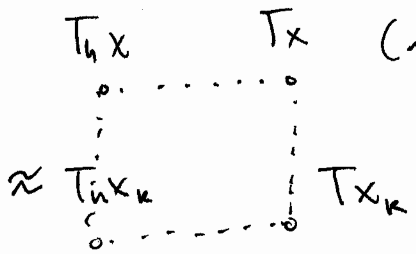
Thm: ($\varepsilon/3$ Theorem)

If $T_n : X \to Y$ (norm, Banach) and $\|T_n\| \le C < \infty$ and X_0 dense in X.

and $T_n x \to Tx$ $\forall x \in X_0$

Then $T_n x \to Tx$ $\forall x \in X$.

Proof:



(must examine the difference).

each step must be bounded by $\varepsilon/3$.

Take $x \in X$, and we know X_0 is dense. So for $\forall \varepsilon > 0$, $\exists x_0$ s.t.

$\|x - x_0\| < \dfrac{\varepsilon}{3C}$.

bounded by C.

Then $\forall n$ $\|T_n x - T_n x_0\| \le \|T_n\| \|x - x_n\| < \dfrac{\varepsilon}{3}$

$\|T\| \le C$ so $\|Tx_0 - Tx\| < \dfrac{\varepsilon}{3}$.

And since $T_n x_0 \to Tx_0$, $\exists N$ s.t. $\forall n > N$, $\|T_n x_0 - Tx_0\| < \dfrac{\varepsilon}{3}$

$\Rightarrow \|T_n x - Tx\| \le \|T_n x - T_n x_0\| + \|T_n x_0 - Tx_0\| + \|Tx_0 - Tx\| < \varepsilon$.

$\Rightarrow$ So $\forall \varepsilon > 0$ $\exists N$ s.t. $\forall n > N$ $\|T_n x - Tx\| < \varepsilon$.

$T_n x_0 \to Tx_0 \Rightarrow \|T_n x_0\| \to \|Tx_0\|$.

restricted to X_0

So $\|Tx_0\| = \lim_{n\to\infty} \|T_n x_0\| \le C \cdot \|x_0\|$. $\Rightarrow$ $\|T|_{X_0}\| \le C$. $\Rightarrow \|T\| \le C$.

<u>Thm</u> (Uniform Boundedness).

<u>If</u> $T_n : \underset{\text{Banach}}{X} \to \underset{\text{Banach}}{Y}$ and $\forall x \in X$ $\exists$ $C_x < \infty$ st $\|T_n x\| \leq C_x$ $\forall n$

<u>Then</u> $\exists$ $C < \infty$ st $\|T_n\| \leq C$.

In particular, <u>if</u> $T_n x \to Tx$ $\forall x$, <u>then</u> $\|T_n\| \leq C < \infty$.

<u>Exercises</u>

1.) $\phi \in C[0,1]$.

$M_\phi : C[0,1] \to C[0,1]$

Def: ~~End~~ ~~Show~~ $M_\phi f = \phi f$.

Show it is bounded. Find $\|M_\phi\|$.

2.) $C[0,a]$ $a < \infty$ $a > 0$.

$J : C[0,a] \to C[0,a]$ <u>Remember:</u> $\|f\|_{C[a,b]} = \max_{x \in [a,b]} |f(x)|$.

$Jf(x) = \int_0^x f(t)\,dt$.

Show it is bounded ~~but not uniformly~~

and compute $\|J\|$.

3.) $Df = f'$ $C'[0,1]$

Show that D cannot be extended to a bounded operator

$C[0,1] \to C[0,1]$.