(Wed) Sept. 17, 2008

- Consider $f_n(x) = x^n$ - what is $\|f_n\|$? in $C[0,1]$

Clearly $\|f_n\| = 1$

$[Df_n](x) = n x^{n-1}$ $\quad \|Df_n\| = n \uparrow \infty$

$\|f_n\| = 1 \quad \|Df_n\| \to \infty$

Hilbert Space:

Inner Product Space: Let H be a vector space.

$H \times H \to \mathbb{F}$ is an inner product, with the following properties:

1. $(\alpha x + \beta y, z) = \alpha(x,z) + \beta(y,z)$ - In Particular: $(\vec{0}, z) = 0$
2. $(x,y) = \overline{(y,x)}$
3. $(x,x) \geqslant 0$
4. $(x,x) = 0 \Rightarrow x = \vec{0}$

e.g. $(x, \alpha y + \beta z) = \bar{\alpha}(x,y) + \bar{\beta}(x,z)$

IS $\|x\| = (x,x)^{1/2}$ a norm?

ℓ^2 : $(x,y) = \sum_{k=1}^{\infty} x_k \bar{y}_k$

$\mathbb{R}^n, \mathbb{C}^n$: $(x,y) = \sum_1^n x_k \bar{y}_k$

$L^2(a,b)$: $(f,g) = \int_a^b f(x)\overline{g(x)}\,dx$

Theorem: (Bynykowsky, Cauchy, Schwartz)

$|(x,y)| \leq \|x\| \, \|y\|$

Proof: $(x+ty, x+ty) = \|x\|^2 + t(y,x) + \bar{t}(x,y) + |t|^2 \|y\|^2$

$= \|x\|^2 + 2\,\mathrm{Re}\{t(y,x)\} + |t|^2 \|y\|^2$

- Assume t and (y,x) are both real, then we have:

$$\|x\|^2 + 2t\,(y,x) + \|y\|^2\, t^2 \quad (*)$$

- For what value of t, is the above minimum?

$$2(y,x) + 2t\|y\|^2 = 0 \quad \rightarrow t = \frac{-(y,x)}{\|y\|^2}$$

Plugging in $t = \frac{-(y,x)}{\|y\|^2}$ into $(*)$, we get:

$$\|x\|^2 - \frac{2(y,x)^2}{\|y\|^2} + \|y\|^2 \frac{(y,x)^2}{\|y\|^4} \geqslant 0 \quad \xrightarrow{\text{multiplying by } \|y\|^2}$$

$$\underline{\|y\|^2 \|x\|^2 - (y,x)^2 \geqslant 0}$$

In the other case: (absence of the previous assumption)

$t = -2 \frac{\overline{(y,x)}}{\|y\|^2}$. Plugging into $(*)$, we arrive @ the following

which is the same result. $\therefore$ $\underline{\|x\|^2 \|y\|^2 \geqslant |(y,x)|^2}$

Proof of "Δ" Inequality:

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(x,y) \leqslant \|x\|^2 + \|y\|^2 + 2|(x,y)|$$

$$\leqslant \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2$$

So: $\|x\| = (x,x)^{1/2}$ is indeed a norm.

The conclusion is that: An Inner product space, is a normed space, but the converse <u>may</u> not be true.

An inner product space, which is also a Banach space is a <u>Hilbert Space</u>.

Def: $x \perp y$ if $(x,y) = 0$ - $\forall \vec{x}$ $\vec{0} \perp \vec{x}$ trivially.

If $\vec{x} \perp \vec{y}$ $\forall \vec{y}$ $\Rightarrow \vec{x} = \vec{0}$

Proof: Take $\vec{y} = \vec{x}$ then $(x,y) = (x,x) = 0 \xrightarrow{\text{by } (4)} \vec{x} = \vec{0}$

Prop: If $\vec{x} \perp \vec{y}$ $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ (Pythogorean Theorem)

Corollary: $x_j \perp x_k$ $k \neq j$ $\rightarrow \left\| \sum_1^n x_k \right\|^2 = \sum_1^n \|x_k\|^2$

Prop: $\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$ ("▱" Identity)

Proof: $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\,(x,y)$ ①

$\|x-y\|^2 = \|x\|^2 + \|y\|^2 + 2\,\underbrace{\mathrm{Re}\,(x,-y)}_{\to\, -2R(x,y)}$ ②

① + ② $= 2\left(\|x\|^2 + \|y\|^2\right)$

Polarization Identities:

1. If $\mathcal{H}$ is a real Hilbert space, then:

$$(x,y) = \frac{1}{2}\left[\|x+y\|^2 - \|x-y\|^2\right]$$

2. If $\mathcal{H}$ is a complex Hilbert space, then:

$$(x,y) = 1/4 \sum_{\alpha = \pm 1, \pm i} \alpha \|x + \alpha y\|^2$$

The proof of (2) is left as an exercise.