Lecture 8

Orthogonal bases

Thm. $\{x_n\}_1^\infty \quad x_j \perp x_k \quad \forall j \neq k$

then $\sum_1^\infty x_k$ converges $\Leftrightarrow$ $\sum_{n=1}^\infty \|x_n\|^2 < \infty$

Moreover $\|\sum_1^\infty x_n\|^2 = \sum \|x_n\|^2$

$S_n = \sum_{k=1}^n x_k$

- $\sum x_k$ converges means $\exists \lim S_n$; $\quad \sum_1^\infty x_k = \lim S_n$

$\sigma_n = \sum_{k=1}^n \|x_k\|^2$

$m > n \qquad \|S_n - S_m\|^2 = \|\sum_{k=n+1}^{m} x_k\|^2 = \sum_{k=n+1}^{m} \|x_k\|^2 = |\sigma_m - \sigma_n|$

So S_n - Cauchy $\Leftrightarrow$ σ_n - Cauchy

$\|S_n - S_m\| < \epsilon \Leftrightarrow |\sigma_n - \sigma_m| < \epsilon^2$

$\lim S_n = S = \sum_1^\infty x_n \Rightarrow \lim \|S_n\|^2 = \|S\|^2$

$\|S_n\| = \left(\sum_{k=1}^n \|x_k\|^2\right)^{1/2}$

$\lim \|S_n\|^2 = \lim_{n\to\infty} \sigma_n = \sum_1^\infty \|x_k\|^2 = \|S\|$

Remark continuity of inner product

If $\lim x_n = x$, then $\forall y \quad \lim (x_n, y) = (x, y)$ (Pf. in the textbook)

$|(x,y)| \le \|x\| \cdot \|y\| \Rightarrow \quad x \longmapsto (x,y)$

$L_y(x) = (x,y)$

$L_y : H \to \mathbb{C}\ (\mathbb{F})$, $\|L_y\| < \infty$

$X_0 \subset \underset{\text{metric}}{X}$ is dense if $\forall\ x \in X\ \exists \{x_n\}_1^\infty\ x_n \in X_0$ s.t. $\lim x_n = X$

Def A metric space X is called separable if $\exists$ countable & dense $X_0 \subset X$

$\mathbb{R}$ uncountable

$\mathbb{Z}, \mathbb{Q}$ - countable

$\mathbb{R}$ - separable

Countable union of countable sets is countable.

$$\underset{(\text{span})}{\mathcal{L}(A)} = \left\{ \sum_{\text{finite}} \alpha_k \vec{a}_k : \alpha_k \in F,\ \vec{a}_k \in A \right\}$$

Thm Normed space X is separable iff $\exists \{x_n\}_1^\infty$ s.t. $\mathcal{L}(\{x_n : n \in \mathbb{N}\})$ is dense.

Pf. ($\Rightarrow$) trivial if $\{x_n\}_1^\infty$ is dense then $\mathcal{L}(\{x_n\}_1^\infty)$ also dense

($\Leftarrow$) $\mathcal{L}_{\mathbb{Q}}(\{x_n\}_1^\infty) = \left\{ \sum_{\text{finite}} \alpha_k \vec{x}_k : \alpha_k \in \mathbb{Q} \right\}$ is dense and countable

($\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$)

Take arb $x \in X$, $\epsilon > 0$

$\exists\ \tilde{\alpha}_k \in \mathbb{R}$ s.t. $\left\| \sum_1^n \tilde{\alpha}_k x_k - x \right\| < \varepsilon/2$

each α_k can be appr. by rationals.

Remark WLOG assume that $\{x_n\}_1^\infty$ is linearly independent.

Thm.

Def. $\{x_n\}_1^\infty$ s.t. $\mathcal{L}\{x_n\}_1^\infty$ is dense in X is called complete (total)

Def. $\{e_n\}_1^\infty$ in $\mathcal{H}$ is called an orthonormal basis (ONB) if $(e_n, e_k) = \delta_{nk} = \begin{cases} 1, & n= \\ 0, & o \end{cases}$

and $\forall\ x \in \mathcal{H}$ can be represented as $x = \sum_1^\infty c_k e_k$

Remark $c_k = (x, e_k)$

$$x = \sum_1^\infty c_k e_k$$

can interchange b/c. inner product $\to$ continuous

$$(x, e_n) = \left(\sum_1^\infty c_k e_k, e_n \right) = \sum_1^\infty c_k (e_k, e_n) = c_n \underbrace{(e_n, e_n)}_{1}$$

$$\left(\lim_{N\to\infty}\sum_{k=1}^{N} c_k e_k, e_n\right) = \lim_{N\to\infty}\sum_{k=1}^{n} c_k (e_k, e_n) = (x, e_n)$$

By continuity $\left(\sum_1^\infty c_k e_k, e_n\right) =$

$$(\lim S_N, e) = \lim (S_N, e)$$

If $\lim x_n = x \Rightarrow \lim (x_n, y) = (x, y)$

<u>Thm.</u> $\{e_n\}_1^\infty$ $(e_n, e_k) = \delta_{k,n}$ is ONB $\Leftrightarrow$ $\{e_n\}_1^\infty$ is complete

<u>Pf.</u> If $\{e_n\}$ is ONB then $\mathcal{L}(\{e_n\}_1^\infty)$ is dense (each x can be appr. by $\sum_1^n c_k e_k$, $c_k = (x, e_k)$)

If $\{e_n\}_1^\infty$ is complete $\Rightarrow \forall \epsilon > 0 \ \exists \ n, \{c_k^n\}_{k=1}^n$ s.t. $\left\|x - \sum_{k=1}^n c_k^n e_k\right\| < \epsilon$

But $\min_{\alpha_1, \dots, \alpha_n \in \mathbb{F}} \left\|x - \sum_{k=1}^n \alpha_k e_k\right\|$ is at $\alpha_k = (x, e_k)$, $k = 1, 2, \dots, n$

<u>why?</u> (check the book)

Then $\left\|x - \sum_{k=1}^n c_k e_k\right\| \le \left\|x - \sum_{k=1}^n c_k^n e_k\right\| < \epsilon$

$c_k = (x, e_k)$

$$\Rightarrow \lim_{n\to\infty} \sum_{k=1}^n c_k e_k = x$$

abstract Fourier series

$c_k = (x, e_k)$

$c_k = \hat{x}(k) = (x, e_k)$