9/24/08

Fourier Series

$L^2(0, 2\pi)$

$e_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}$ where $\cancel{N} \in \mathbb{Z}$

$\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal system.

ie $\langle e_n, e_k \rangle = \delta_{nk}$

$$\langle e_n, e_k \rangle = \int_0^{2\pi} \frac{1}{\sqrt{2\pi}} e^{int} \overline{\frac{1}{\sqrt{2\pi}} e^{ikt}}\, dt = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{e^{int} e^{-ikt}}_{e^{i(n-k)t}}\, dt$$

$$= \frac{1}{2\pi} \frac{1}{i(n-k)} e^{i(n-k)t} \Big|_0^{2\pi} = 0 \qquad n \neq k$$

$$= \frac{1}{2\pi} \int_0^{2\pi} 1\, dt = 1 \qquad n = k \quad \Rightarrow \text{ONB.}$$

$\frac{1}{\sqrt{2\pi}}$, $\frac{1}{\sqrt{\pi}} \cos nt$, $\frac{1}{\sqrt{\pi}} \sin nt$. $n = 1, 2, 3 \ldots$

$$\frac{1}{\sqrt{2}}(e_n + e_{-n}) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2\pi}}(e^{int} + e^{-int})\right)$$

$$= \frac{e^{int} - e^{-int}}{2\sqrt{\pi}} = \frac{1}{\sqrt{\pi}} \cos nt.$$

$$\frac{1}{\sqrt{2}\, i}(e_n - e_{-n}) = \frac{1}{i\sqrt{2}} \frac{1}{\sqrt{2\pi}}(e^{int} - e^{-int})$$

$$= \frac{1}{\sqrt{\pi}}\left(\frac{e^{int} - e^{-int}}{2i}\right) = \frac{1}{\sqrt{\pi}} \sin nt.$$

$\mathcal{L}\{e_k : -n \leq k \leq n\}$

$= \mathcal{L}(1, \sin k\theta, \cos k\theta) \quad k = 1, 2, \dots n.$

) Two ways to write down the same thing.

<u>Motivation</u>:

<u>Heat Equation</u>

$$\begin{cases} u_t - k u_{xx} = 0 & u(t,x),\ t \in [0,\infty) \\ 0 \leq x \leq \pi & u|_{t=0} = u_0. \quad u(t,0) = u(t,\pi) = 0. \quad \text{Dirichlet} \end{cases}$$

or

$u_x(t,0) = u_x(t,\pi) = 0.$ Neuman.

<u>Solutions</u>: $(\sin(nx))\, e^{-kn^2 t} \quad n \geq 1 \quad (D)$

$(\cos nx)\, e^{-kn^2 t} \quad n \geq 0 \quad (N).$

$u_0(x) = \sum c_k \sin kx$

$u(t,x) = \sum (c_j \sin jx)\, e^{-jn^2 t}$

Extend u_0 to $[-\pi, \pi]$ $\quad u_0(-x) = -u_0(x)$. and decompose in F.S.

$a_0 + \sum_{n \geq 1} a_n \cos nx + b_n \sin nx$

$a_n = 0$

$L^2(0, 2\pi) \sim L^2_{2\pi}$ periodic $\sim L^2(\mathbb{T}) \sim L^2(\mathbb{R}/2\pi\mathbb{Z})$.

unit circle $\in \mathbb{C}$

$\|f\|^2_{L^2_{2\pi}} = \int_a^{a+2\pi} |f(t)|^2\, dt.$

Orth. proj.

$f \in L^2_{2\pi} \to P_n f = \sum_{k=-n}^{n} \langle f, e_k \rangle e_k. \qquad \langle f, e_k \rangle = \int_a^{a+2\pi} f(s) e^{-iks} \frac{1}{\sqrt{2\pi}}\, ds.$

$(P_n f)(t) = \sum \left(\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(s) e^{-iks}\, ds \right) \frac{1}{\sqrt{2\pi}} e^{ikt} = \int_0^{2\pi} f(s) \frac{1}{2\pi} \left(\sum_{k=-n}^{n} e^{ik(t-s)} \right) ds$

$t - s = x$

$$\sum_{k=-n}^{n} e^{ikx} = \frac{e^{-inx} - e^{i(n+1)x}}{1 - e^{ix}} = \frac{e^{i(n+1/2)x} - e^{-i(n+\frac{1}{2})x}}{e^{ix/2} - e^{-ix/2}} \cdot \left(\frac{\frac{1}{2i}}{\frac{1}{2i}} \right)$$

$$= \frac{\sin((n+\frac{1}{2})x)}{\sin(\frac{x}{2})} := D_n(x) \Rightarrow \text{Dirichlet Kernel.}$$

$$(P_n f)t = \int_a^{a+2\pi} f(s) D_n(t-s) \frac{ds}{2\pi}. \Rightarrow \text{Convolution}$$

$$(f * g)t = \int_0^{2\pi} f(s) g(t-s)\, ds$$

$$P_n f = \frac{1}{2\pi}(f * D_n). \qquad (f * g)t = \int_0^{\infty} f(s) g(t-s)\, ds$$

Proposition

$$\|D_n\|_{L^1_{2\pi}} \sim \ln(n)$$

$$\Rightarrow \exists\, c, C \quad \text{s.t.} \quad \int_{-\pi}^{\pi} |D_n(t)|\, dt \leq C \ln(n)$$
$$\geq c \ln(n).$$

Warm Up

Estimate $\displaystyle\sum_{n=1}^{10^{23}} \frac{1}{n}$

Try comparing to $\displaystyle\int_1^{10^{23}} \frac{1}{x}\, dx$.