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Convergence of Fourier Series

In $L^2_{2\pi}$ is easy

$$\hat{f}(n) := (f, e_n)$$

By Bessel's Inequality $\quad \sum |\hat{f}(n)|^2 = \|f\|^2_{L^2_{2\pi}}$

$$\Rightarrow \sum \hat{f}(n) e_n = \sum (f, e_n) e_n \quad \text{converges.}$$

Thm: $\exists\ f \in C_{2\pi}$ s.t. $(P_n f)(0)$ does not converge.

Pf: $\quad L_n : C_{2\pi} \longrightarrow \mathbb{C}$ or $\mathbb{R}$

$$L_n f := (P_n f)(0)$$

$$L_n f = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) D_n(-s)\, ds$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) D_n(s)\, ds \qquad \text{because } D_n \text{ is even}$$

Claim: $\|L_n\| = \frac{1}{2\pi} \|D_n\|_{L^1}$

Indeed if $f \in C_{2\pi}$

$$\left| \int_{-\pi}^{\pi} f(s) D_n(s)\, ds \right| \leq \int_{-\pi}^{\pi} \|f\|_{C_{2\pi}} |D_n(s)|\, ds$$

$$= \|D_n\|_{L^1} \|f\|_{C_{2\pi}}$$

$$\Rightarrow \|L_n\| \leq \|D_n\|_{L^1}$$

Take $f(s) = \operatorname{sign} D_n(s)$ - we can approximate this f close enough with a continuous function to work out.

$|f(s)| \leq 1 \qquad \|f\| \leq 1$

$$\int_{-\pi}^{\pi} D_n(s) f(s)\, ds = \int_{-\pi}^{\pi} |D_n(s)|\, ds$$

So $\|L_n\| \geq \|D\|_{L^1}$

By Uniform Boundedness Principle if $L_n f$ is bdd $\forall f \in C_{2\pi}$

then $\|L_n\| \leq C$

But $\|L_n\| = \|D_n\|_{L^1} \geq c \ln n \to \infty$, so $\exists f \in C_{2\pi}$

$(P_n f)(0)$ is unbounded.

Convolution

Lemma: $f * g(t) = \int_0^{2\pi} f(s)\, g(t-s)\, ds$ is commutative for f, g 2π-periodic

Pf: $f*g(t) = \int_0^{2\pi} f(s)\, g(t-s)\, ds$ define $t-s = \tilde{s}$ then $s = t - \tilde{s}$ and $ds = -d\tilde{s}$

$$= -\int_t^{t-2\pi} f(t-\tilde{s})\, g(\tilde{s})\, d\tilde{s} = \int_0^{2\pi} f(t-\tilde{s})\, g(\tilde{s})\, d\tilde{s} = g*f(t)$$

Lemma: $T: C_{2\pi} \to C_{2\pi}$, $Tf = F*f$, $F \in C_{2\pi}$

Then $\|T\| = \|F\|_{L^1_{2\pi}}$

Pf: $(\leq)$ $|F*f(t)| = \left| \int_0^{2\pi} F(s)\, f(t-s)\, ds \right| \leq \|f\|_{C_{2\pi}} \int_{-\pi}^{\pi} |F(s)|\, ds = \|f\|_{C_{2\pi}} \|F\|_{L^1_{2\pi}}$

so $\|T\| \leq \|F\|_{L^1_{2\pi}}$

$(\geq)$ $f = \operatorname{sign} F$ or $\frac{|F|}{F}$ for complex, approximation with continuous function again.

Fejer Means and Kernel

For a divergent series consider average of partial sums: $\frac{S_1 + S_2 + \dots + S_n}{n}$

Now consider $F_n f = \frac{P_0 f + P_1 f + \dots + P_{n-1} f}{n}$

So $F_n f = \frac{1}{2\pi} F_n * f$ where $F_n(x) = \frac{D_0(x) + D_1(x) + \dots + D_{n-1}(x)}{n}$

$$F_n(x) = \frac{1}{n} \frac{(\sin \frac{nx}{2})^2}{(\sin \frac{x}{2})^2} \geq 0$$

$\frac{1}{2\pi} \int_{-\pi}^{\pi} |F_n(x)|\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(x)\, dx = 1$, because $\frac{1}{2\pi} \int_{-\pi}^{\pi} D_k(x)\, dx = 1$

$$D_n(x) = \sum_{k=-n}^{n} e^{ikx}$$

$$\int_{-\pi}^{\pi} e^{ikx}\, dx = \begin{cases} 0 & k \neq 0 \\ 2\pi & k = 0 \end{cases}$$