Lecture 12

$\exists\ f \in C_{2\pi}$ s.t. $(P_n f)(0)$ unbounded

$\|D_n\|_{L^1} \sim \ln n$

Consider Fejer means $\sigma_n = \frac{S_1 + \dots + S_n}{n}$

For Fourier series: $F_n = \frac{P_0 + P_1 + \dots + P_{n-1}}{n}$

$P_K f = \frac{1}{2\pi} D_K * f$

$F_n f = \frac{1}{2\pi} F_n * f$, where $F_n = \frac{1}{n}\sum_{K=0}^{n-1} D_K$

Claim

$$F_n(t) = \frac{1}{n}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2$$

Pf

$$n F_n(t) = \sum_{K=0}^{n-1} \frac{\sin(K+\frac{1}{2})t}{\sin\frac{t}{2}} \Rightarrow F_n(t)\, n \left(\sin\frac{t}{2}\right)^2 = \sum_{K=0}^{n-1} \sin(K+\tfrac{1}{2})t \, \sin\frac{t}{2}$$

$$= \sum_{K=0}^{n-1} \frac{1}{2}\left[\cos Kt - \cos(K+1)t\right]$$

We know that $\cos\alpha - \cos\beta = -2\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}$

$$\Rightarrow F_n(t)\, n \left(\sin\frac{t}{2}\right)^2 = \frac{1}{2}(1 - \cos nt) = \sin^2\frac{nt}{2}$$

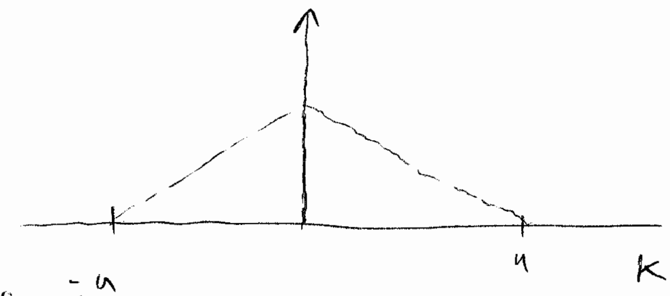
$$\boxed{\frac{1-\cos 2\alpha}{2} = \sin^2\alpha}$$

* $\sin(K+\frac{1}{2})t = \operatorname{Im} e^{i(K+\frac{1}{2})t}$

$F_n(t) \geq 0$

$$\int_0^{2\pi} |F_n(t)|\, dt = \int_0^{2\pi} F_n(t)\, dt = 2\pi \quad \left(\text{b/c. } \int_0^{2\pi} D_n(t)\, dt = 2\pi\right)$$

$$F_n(t) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right) e^{ikt}$$



<u>Thm</u> let $f \in C_{2\pi}$

Then $\frac{1}{2\pi} F_n * f \rightrightarrows f$

converges uniformly

bc $\int F_n(s)\,ds = 2\pi$

<u>Pf</u> $\left|\frac{1}{2\pi} F_n * f(t) - f(t)\right| = \left|\frac{1}{2\pi} \int \left[f(t-s) - f(t)\right] F_n(s)\,ds\right| \overset{**}{=}$

$$\boxed{(F_n * f)(t) = f * F_n(t) = \int_{-\pi}^{\pi} f(t-s)\, F_n(s)\,ds}$$

- f is uniformly continuous
  $\forall \epsilon > 0 \ \exists \delta \ \forall s, \ |s| < \delta \quad \forall t \quad |f(t-s) - f(t)| < \frac{\epsilon}{2}$

- periodic f-tions → unif. cont. $\quad f \in C_{2\pi} \Rightarrow f \in C[0, 2\pi]$

This doesn't work with Dirichlet kernel

$$\overset{**}{=} \frac{1}{2\pi}\left|\int_{-\delta}^{\delta} \dots + \int_{[-\pi,\pi]\setminus(-\delta,\delta)} \dots\right| \le \underbrace{\frac{\epsilon}{2}}_{\ge |f(t-s)-f(t)|} \cdot \frac{1}{2\pi} \int_{-\delta}^{\delta} F_n(t)\,dt + 2\|f\|_{C_{2\pi}} \frac{1}{2\pi} \int_{[-\pi,\pi]\setminus(-\delta,\delta)} F_n(t)\,dt$$

$$\le \frac{\epsilon}{2} + 2\|f\|_{C_{2\pi}} \int_{[-\pi,\pi]\setminus(-\delta,\delta)} F_n(t)\,dt$$

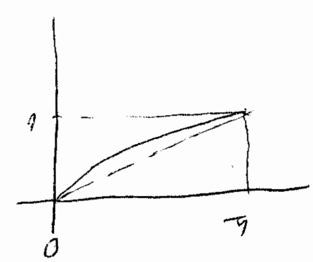
<u>OBS</u> $\forall \delta > 0$

$F_n(t) \rightrightarrows 0$ on $[-\pi, \pi] \setminus (-\delta, \delta)$

$\delta \le |t| \le \pi$

$\left|\sin \frac{t}{2}\right| \ge \frac{\delta}{\pi}$

$0 \le F_n(t) \le \frac{1}{n}\left(\frac{\pi}{\delta}\right)^2$



$\exists N$ s.t. $\forall n > N \quad |F_n(t)| < \frac{\epsilon}{4\|f\|_{C_{2\pi}}}$ on $[-\pi, \pi] \setminus (-\delta, \delta)$

$$\overset{**}{\le} \frac{\epsilon}{2} + 2\|f\| \cdot \frac{1}{2\pi} \frac{\epsilon}{4\|f\|} \cdot 2\pi \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Corollaries

1. Trig. polynomials are dense in $C_{2\pi}$

$$\sum_{-N}^{N} a_k e^{ikt}$$

Pf $\frac{1}{2\pi} F_n * f$ is a trig. polynomial

2. Trig. polynomials are dense in $L^2_{2\pi}$ so $\{e_n\}_{-\infty}^{\infty}$ $e_n(t) = \frac{e^{int}}{\sqrt{2\pi}}$ is ONB in $L^2_{2\pi}$

$\mathrm{Clos}\, X = \{\text{set of all } \lim_{n\to\infty} x_n,\ x_n \in X\}$

Matryoshka principle

Pol. - trig. poly.

$\mathrm{Clos}_{C_{2\pi}} \mathrm{Pol.} = C_{2\pi}$

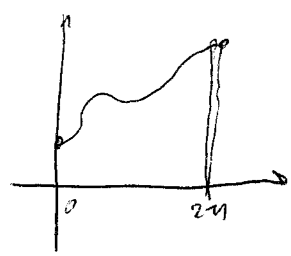
$\mathrm{Clos}_{L^2_{2\pi}} \mathrm{Pol.} \supset C_{2\pi}$

$\mathrm{Clos}_{L^2_{2\pi}} C_{2\pi} \supset C[0,2\pi]$

$\mathrm{Clos}_{L^2} C[0,2\pi] = L^2[0,2\pi] \sim L^2_{2\pi}$

$f \in C_{2\pi} \Rightarrow f(0) = f(2\pi)$

for cont. f. this is not necessary



$(\mathrm{Clos}\,\mathrm{Clos}\, X = \mathrm{Clos}\, X)$

$f \in L^2[0,2\pi]$

$\varepsilon/3 >$

$f_1 \in C[0,2\pi]$

$\varepsilon/3 >$

$f_2 \in C_{2\pi}$

$\varepsilon/3 >$ $f_3 \in Pol$