10/1

Trigonometric polynomials are dense in $C_{2\pi}$ $\Rightarrow$ dense in $L^2_{2\pi}$

$\Rightarrow$ dense in $L^p_{2\pi}$ $\forall p$ $1 \le p < \infty$

by Matryoshka Principle.

Stone-Weierstrauss Thm: $f \in L^2_{2\pi}$ then $f(t) = \sum \hat{f}(n) e^{int}$ converges in L^2

$\hat{f}(n) = \int_0^{2\pi} f(t) e^{-int} \frac{dt}{2\pi}$

Notation: $\|f\|_{L^p_{2\pi}} := \left(\int |f(t)|^p \frac{dt}{2\pi} \right)^{1/p}$

$$f * g = \int_0^{2\pi} f(s) g(t-s) \frac{ds}{2\pi}$$

Convergence of Fejer Means in L^p

Thm: $f \in L^p_{2\pi}$, $1 \le p < \infty$, then $F_n * f \to f$ in L^p, i.e., $\|F_n * f - f\|_p \to 0$

Lemma (Convolution lemma): $\|F_n * f\|_p \le \|F\|_1 \|f\|_p$

Pf: Recall Resonance Lemma: $\|F*f\|_p = \sup \left\{ \int F*f(t)\, g(t) \frac{dt}{2\pi} : g \in L^q,\ \|g\|_q \le 1,\ \frac{1}{p} + \frac{1}{q} = 1 \right\}$

$$\Rightarrow \left| \int_0^{2\pi} F * f(t)\, g(t) \frac{dt}{2\pi} \right| \le \int_0^{2\pi}\int_0^{2\pi} |F(s) f(t-s) g(t)| \frac{ds}{2\pi} \frac{dt}{2\pi}$$

Since we have positive function, Tonelli Thm lets us change order of integration.

$$= \int_0^{2\pi} \underbrace{\left(\int_0^{2\pi} |f(t-s) g(t)| \frac{dt}{2\pi} \right)}_{\le \|f\|_p \|g\|_q \text{ by Hölder}} |F(s)| \frac{ds}{2\pi}$$

$$\le \int_0^{2\pi} \|f\|_p \|g\|_q |F(s)| \frac{ds}{2\pi} \le \|F\|_1 \|f\|_p$$

Pf of Thm: $f \in$ Trig Polynomial

$F_n * f \rightrightarrows f$ uniformly so converges in L^p

Poly are dense in L^p $1 \le p < \infty$ so $f \mapsto F_n * f$ are uniformly bounded by $\|F_n\|_1 = 1$

$\Rightarrow$ by $\varepsilon/3$ Theorem we have convergence in L^p.

Cor (Uniqueness Thm): $f \in L^1$, $\hat{f}(n) = 0 \Rightarrow f \equiv 0$

Pf: If $\hat{f}(n) = 0 \ \forall n \Rightarrow F_n * f = 0 \ \forall n$

Remark: $f \in L^1$ $\quad \sum \hat{f}(n) e^{int}$ $\quad \hat{f}(n) = \int f(t) e^{-int} \frac{dt}{2\pi}$

Lemma: $\widehat{f*g}(n) = \hat{f}(n)\hat{g}(n)$

Pf Sketch: Check for $f = e^{int}$ $\quad g = e^{ikt}$

$\Downarrow$

$f, g \in$ Poly $\Rightarrow$ extend to L^p by $\varepsilon/3$ Thm

Remark: $f \in L^p$ $\quad p \in (1, \infty)$ then $\sum_{n \in \mathbb{Z}} \hat{f}(n) e^{int}$ converges to f in L^p (will not prove)

Pointwise Convergence and Riemann-Lebesgue Lemma

Riemann-Lebesgue Lemma: $f \in L^1$, $\hat{f}(n) \to 0$ as $n \to \pm\infty$

Cor: $f \in L^1_{2\pi}$, $\Rightarrow$ $\int_0^{2\pi} f(t)\cos(nt)\,dt \to 0$, $\int_0^{2\pi} f(t)\sin(nt)\,dt \to 0$ as $n \to \pm\infty$

Pf: $\cos(nt) = \dfrac{e^{int} + e^{-int}}{2}$ $\quad \sin(nt) = \dfrac{e^{int} - e^{-int}}{2}$

Pf of Lemma: $T_n : L^1 \to \mathbb{C}$ $\quad n \in \mathbb{Z}$ $\quad T_n(f) := \hat{f}(n) = \int_0^{2\pi} f(t) e^{-int} \frac{dt}{2\pi}$

$|T_n f| \leq \|f\|_1 \quad \Rightarrow \quad \|T_n\| \leq 1$

If $f \in \text{Poly} \Rightarrow \lim_{n \to \infty} T_n f = 0$

Poly are dense in $L^1_{2\pi}$

$\Rightarrow T_n f \to 0$ as $n \to \pm\infty$

Thm (Dini's Condition): Let $f \in L^1$ $\quad \int_{-\pi}^{\pi} \left| \frac{f(s_0 - s)}{s} \right| ds < \infty$

Then $(P_n f)(s_0) \to 0$

Remark: $\int_{-\pi}^{\pi}$ can be replaced by $\int_{-\delta}^{\delta}$ because the only singularity is 0

If $\int_{-\pi}^{\pi} \left| \frac{f(s_0 - s) - f(s_0)}{s} \right| ds < \infty \Rightarrow P_n f(s_0) \to f(s_0)$

because $P_n(f - f(s_0)) = P_n f - f(s_0) \xrightarrow[\substack{s = s_0 \\ n \to \pm\infty}]{} 0$

Def: f is a Lipschitz function, $f \in \text{Lip}$ if $\exists\, C$ s.t.

$f(x) - f(y) \leq C|x - y|$

Note: $C^1_{2\pi} \subset \text{Lip}$ by mean value thm

If $f \in \text{Lip} \Rightarrow \forall s_0 \quad \int_{-\pi}^{\pi} \left| \frac{f(s_0 - s) - f(s)}{s} \right| ds < \infty$

Pf of Thm: $(P_n f)(s_0) = (D_n f)(s_0)$

$$D_n(s) = \frac{\sin[(n + \frac{1}{2})s]}{\sin \frac{s}{2}} = \frac{\sin\frac{s}{2} \cos ns + \cos\frac{s}{2} \sin ns}{\sin\frac{s}{2}}$$

$$= \int_{-\pi}^{\pi} f(s_0 - s) D_n(s) \frac{ds}{2\pi}$$

$$= \underbrace{\int_{-\pi}^{\pi} f(s_0 - s) \cos ns \frac{ds}{2\pi}}_{\downarrow\, 0} + \int_0^{2\pi} \underbrace{\frac{f(s_0 - s) \cos \frac{s}{2}}{\sin \frac{s}{2}}}_{\in L^1} \sin ns$$

because $\frac{f(s_0 - s)}{s}$ is integrable

$\frac{s}{\sin\frac{s}{2}}$ is bounded

$\cos\frac{s}{2}$ is bounded