Lecture 14

$f \in L^2_{2\pi}$ $\qquad \frac{dt}{2\pi}$

$$f(t) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{int}, \qquad \hat{f}(n) = \int_{-\pi}^{\pi} f(t) e^{-int} \frac{dt}{2\pi}$$

$$\|f\|^2_{L^2_{2\pi}} = \sum |\hat{f}(n)|^2$$

$$\int_{-\pi}^{\pi} |f(t)|^2 \frac{dt}{2\pi}$$

$f \in L^p$, $\|f\|_{L^p} = ?$

$\exists f \in L^p$; $\{\sigma_n\}_{n\in\mathbb{Z}}$ $\sigma_n = \pm 1$ s.t. $g = \sum \sigma_n \hat{f}(n) e^{int} \notin L^p$

Thm. Let $\{a_k\}$: $\sum |a_k|^2 < \infty$
Then $\exists$ $f \in C_{2\pi}$ s.t. $|\hat{f}(n)| \geq |a_n|$

Sobolev spaces $W^{1,2}_{2\pi}$

$f \in C^1_{2\pi}$ $\qquad \|f\|^2 = \int_{-\pi}^{\pi} |f(t)|^2 \frac{dt}{2\pi} + \int_{-\pi}^{\pi} |f'(t)|^2 \frac{dt}{2\pi} = \|f\|^2_{L^2_{2\pi}} + \|f'\|^2_{L^2(2\pi)}$

Take the completion $\qquad \|f\|^2_{W^{1,2}_{2\pi}} = \sum_{n\in\mathbb{Z}} (1+|n|)|\hat{f}(n)|^2, \qquad \hat{f}(n) = \int_{-\pi}^{\pi} f(t) e^{-int} \frac{dt}{2\pi}$

$$\|f\|^2_{L^2_{2\pi}} = \sum |\hat{f}(n)|^2$$

$$f' = \sum_{n\in\mathbb{Z}} in \hat{f}(n) e^{int}$$

$$\|f'\|^2 = \sum n^2 |\hat{f}(n)|^2$$

Thm $\mathcal{H}$ - separable iff $\exists$ countable ONB

$\{e_n\}_{n=1}^{\infty}$

$\exists$ countable ONB $\Rightarrow$ $f = \sum_n (f, e_n) e_n$

$$\|f\|_{\mathcal{H}} = \| \{(f, e_n)\}_{n=1}^{\infty} \|_{\ell^2} = \left(\sum (f, e_n)^2 \right)^{1/2}$$ [redacted]

e.g.: $L^2_{2\pi}$ $\frac{dt}{2\pi}$ $e_n = e^{int}$

Non-separable Hilbert space

$\mathcal{H} = \{ f$ on $\mathbb{R}$, $f(x) \neq 0$ at countably many points$\}$

s.t. $\|f\|^2 = \sum_{x \in \mathbb{R}} |f(x)|^2 < \infty$

$(f, g) = \sum_{x \in \mathbb{R}} f(x) \overline{g(x)}$; by Hölder $\sum |f(x) \overline{g(x)}| \leq \left(\sum |f(x)|^2 \right)^{1/2} \cdot \left(\sum |g(x)|^2 \right)^{1/2}$

let $\delta_x(t) = \begin{cases} 1 & t = x \\ 0 & t \neq x \end{cases}$ $\rightarrow$ uncountable ONB

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f s.t. $\lim_{T \to +\infty} \frac{1}{2\pi} \int_{-T}^{T} |f(x)|^2 dx < \infty$

$\mathcal{H}$, $E \subset \mathcal{H}$, $x \in \mathcal{H}$

closed subspace

$\exists!\, y \in E$ s.t. $x - y \perp E$

(not.:) $y = P_E x$

E - subspace

$E^{\perp} = \{ x \in \mathcal{H} : x \perp E \} = \{ x \in \mathcal{H} : (x, y) = 0 \quad \forall y \in E \}$

$(E^{\perp})^{\perp} = E$

Linear functionals on $\mathcal{H}$

$X \longrightarrow \mathbb{C}$

Thm. (Riesz Representation Thm.)

If φ is a linear functional on $\mathcal{H}$, then $\exists!\, y \in \mathcal{H}$ s.t $\varphi(x) = (x, y)\ \forall x \in \mathcal{H}$

Moreover, $\|\varphi\| = \|y\|_{\mathcal{H}}$

Remark for $y \in \mathcal{H}$ define $\varphi = \varphi_y$ by $\varphi_y(x) = (x, y)$

Clearly $\underset{\substack{\text{linear}\\ \text{bounded}}}{\varphi} : \mathcal{H} \to \mathbb{C}$

$$\left.\begin{array}{l} |\varphi_y(x)| = |(x,y)| \underset{\text{C.B.S}}{\leq} \|x\| \cdot \|y\|, \text{ so } \|\varphi_y\| \leq \|y\| \\ \text{On the other hand, } |\varphi_y(y)| = \|y\|^2 = \|y\| \cdot \|y\| \Rightarrow \|\varphi_y\| \geq \|y\| \end{array}\right\} \Rightarrow \|\varphi_y\| = \|y\|$$

Pf consider $E = \ker \varphi = \{x \in \mathcal{H} : \varphi(x) = 0\}$

$E^{\perp}$

Claim: $\dim E^{\perp} = 1$ (if $\varphi \neq 0$)

$\varphi \neq 0 \Rightarrow E \subsetneq \mathcal{H} \Rightarrow E^{\perp} \neq \{0\} \Rightarrow \dim E^{\perp} \geq 1$

- take $x_1, x_2 \in E^{\perp}$; $\exists\, \alpha_1, \alpha_2$ s.t. $\underbrace{\alpha_1 \varphi(x_1) + \alpha_2 \varphi(x_2)}_{\shortparallel} = 0$, $|\alpha_1| + |\alpha_2| \neq 0$

$$\varphi(\alpha_1 x_1 + \alpha_2 x_2) = 0 \Rightarrow \left.\begin{array}{r}\alpha_1 x_1 + \alpha_2 x_2 \in E \\ \in E^{\perp}\end{array}\right\} \Rightarrow \alpha_1 x_1 + \alpha_2 x_2 = 0$$

$\Downarrow$

any 2 vectors lin dep.

$\Downarrow$

$\boxed{\dim E^{\perp} \leq 1}$

Pick $\vec{x}_0 \in E^{\perp}$, $\vec{x}_0 \neq \vec{0}$. Take y s.t $(\vec{x}_0, \vec{y}) = \varphi(x_0)$

$$\vec{y} = \frac{\varphi(x_0)}{\|\vec{x}_0\|^2} \vec{x}_0$$

Take $x \in \mathcal{H}$. $x = x_1 + x_2$ s.t. $x_2 = P_E x$, $x_1 = x - x_2 \in E^{\perp} \Rightarrow x_1 = \alpha x_0 \Rightarrow x = \alpha x_0 + x_2$

$$\varphi(x) = \varphi(\alpha x_0) + \underset{\substack{\| \\ 0}}{\varphi(x_2)} = \alpha \varphi(x_0)$$

$$(x,y) = (\alpha x_0 + x_2, y) = \alpha(x_0, y) + \underset{\substack{\| \\ 0}}{(x_2, y)} = \alpha \varphi(x_0)$$