Adjoint Operators

$A : \mathcal{H}_+ \to \cancel{\mathcal{H}_-}\ \mathcal{K}$ (Acts from 1 Hilbert space to another).

~~For a vector~~ $\cancel{x \in \mathcal{H}, \quad L_x : \mathcal{K} \to \mathbb{C}}$.

$\cancel{L_x(y) = (Ax, y) \quad y \in \mathcal{K}}$.

$\cancel{|L_x(y)| = |\langle Ax, y\rangle| \le \|Ax\|\|y\| \le \|A\|\|x\|\|y\|}$.

$\cancel{\Rightarrow L}$ ~~is bounded linear functional on~~ $\cancel{\mathcal{K}}$?

~~So~~ $\cancel{\exists!\ z \in \mathcal{K}}$ ~~s.t.~~ $\cancel{\langle Ax}$

Fix $y \in \mathcal{K}$ $\quad L_y(x) = \langle Ax, y\rangle$

$L_y(x) : \ |\langle Ax, y\rangle| \le \|Ax\|\,\|y\| \le \|A\|\,\|x\|\cdot\|y\|$.

So L_y is bounded.

So $\exists!\ x^* \in \mathcal{H}$ s.t. $L_y(x) = (x, x^*)\ \forall x \in \mathcal{H}$.

$\Rightarrow \langle Ax, y\rangle = (x, x^*)$. $\forall x \in \mathcal{H}$.

Define $x^* \equiv A^* y$.

<u>Proposition</u>: A^* is bounded linear operator $\mathcal{K} \to \mathcal{H}$.

Note: $\langle Ax, y\rangle = \langle x, A^* y\rangle$

<u>A^*</u> is linear.

$\langle Ax, \alpha y_1 + \beta y_2\rangle = \bar{\alpha}\langle Ax, y_1\rangle + \bar{\beta}\langle Ax, y_2\rangle$

$= \langle x, A^*(\alpha y_1 + \beta y_2)\rangle$

$= \bar{\alpha}\langle x, A^* y_1\rangle + \bar{\beta}\langle x, A^* y_2\rangle = \langle x, \alpha A^* y_1\rangle + \langle x, \beta A^* y_2\rangle$

$= \langle x, \alpha A^* y_1 + \beta A^* y_2\rangle$.

Prop: A^* is bounded and $\|A^*\| = \|A\|$

Lemma: $\|A\| = \sup\{|\langle Ax, y\rangle| : x \in \mathcal{H}, y \in \mathcal{K}, \|y\|\|x\| \le 1\}$

Pf

$\|A\| = \sup\{\|Ax\| : x \in \mathcal{H}, \|x\| \le 1\}$

$\|Ax\| = \sup\{\langle Ax, y\rangle : x \in \mathcal{H}, y \in \mathcal{K}, \|y\| \le 1\}$

$\langle Ax, y\rangle = \overline{(A^* y, x)}$

Example $A: \mathbb{C}^n \to \mathbb{C}^m$ $\{a_{jk}\}_{j=1}^{m}{}_{k=1}^{n}$

$A^* = \overline{A^T}$ → Hermitian Conjugate.

If A acts from $\mathcal{H} \to \mathcal{K}$
then A^* acts from $\mathcal{K} \to \mathcal{H}$, then AA^* and A^*A exist.

Example Consider $L_2(a,b) = \mathcal{H}$

ϕ is bounded M_ϕ is multiplication operator $\mathcal{H} \to \mathcal{H}$.

$M_\phi(f) = \phi \cdot f$.

Claim $M_\phi^* = M_{\bar\phi} f$. $\langle M_\phi f, g\rangle = \int_a^b \phi f \bar g \, dt = \int_a^b f \, \overline{\bar\phi g} \, dt = \langle f, M_{\bar\phi} g\rangle$

Define: $A: \mathcal{H} \to \mathcal{H}$ is called
self adjoint if $A^* = A$. $\boxed{\text{self adjoint} = \text{hermitian}}$ $= M_{\bar\phi} g$

$\boxed{\text{adjoint} = \text{hermitian conjugate}}$

Defin: $\mathcal{U}: \mathcal{H} \to \mathcal{K}$ is called unitary if $\mathcal{U}^{-1} = \mathcal{U}^*$.

If $|\phi| = 1$ then $M_\phi : L^2(a,b) \to L^2(a,b)$ is unitary.

Define :

$N : \mathcal{H} \to \mathcal{H}$ is called normal if $N^* N = N N^*$

Example

$\mathcal{F} : L^2_{2\pi} \to \ell^2(\mathbb{Z})$. $\mathcal{F} f = \{\hat{f}(n)\}_{n \in \mathbb{Z}}$.

F is a ~~uni~~ unitary operator.

$F^{-1}\{a_n\} = \sum_{n \in \mathbb{Z}} a_n e^{int}$

If $\mathcal{U} : \mathcal{H} \to \mathcal{K}$ s.t. $\|\mathcal{U} f\| = \|f\|$ $\forall f \in \mathcal{H}$ and $\mathcal{U}$ is invertible, then $\mathcal{U}$ is unitary

Def. $\mathcal{U}$ s.t $\|\mathcal{U} f\| = \|f\|$ $\forall f$ is called isometry.

Invertible isometry is same as unitary.

Pf. $\|\mathcal{U} f\|^2 = \|f\|^2 = \langle \mathcal{U} f, \mathcal{U} f \rangle = \langle f, f \rangle$

$= \langle \mathcal{U}^* \mathcal{U} f, f \rangle = \langle f, f \rangle \quad \forall f.$

Use Polarization Identity :

$$\langle A f, g \rangle = \frac{1}{4} \sum_{\alpha = \pm 1, \pm i} \alpha \langle A(f + \alpha g), (f + \alpha g) \rangle$$

$$\langle \mathcal{U}^* \mathcal{U} f, g \rangle = \langle f, g \rangle \quad \forall f, g$$

$$\Rightarrow \mathcal{U}^* \mathcal{U} f = f \quad \forall f \Rightarrow \mathcal{U}^* \mathcal{U} = e$$

Since $\mathcal{U}$ is invertible, $\mathcal{U}^* = \mathcal{U}^{-1}$. $\Rightarrow$ Unitary.