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$$f(x) = x \qquad \hat{f}(0) = \int_{-\pi}^{\pi} x \frac{dx}{2\pi} = 0$$

$$\int_{-\pi}^{\pi} x e^{-inx} dx = -\frac{1}{in}\int_{-\pi}^{\pi} x \, d e^{-inx}$$

$$= -\frac{1}{in}\left(x e^{-inx} \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} e^{-inx} dx \right)$$

$$= \frac{\pi}{-in}\left(e^{-in\pi} + e^{in\pi}\right) \cancel{- \frac{1}{(in)^2} e^{-inx} \Big|_{-\pi}^{\pi}} \qquad \leftarrow \text{periodic}$$

$$= \frac{\pi}{-in} 2\cos \pi n = \frac{2\pi}{-in} \cos \pi n$$

$$\Rightarrow |\hat{f}(n)| = \frac{1}{|n|}$$

$$\cancel{= \int_0^{\pi} t e^{int} dt}$$

~~take $x = t$~~ $$\cancel{= \int_0^{\pi} x e^{inx} dx = \frac{1}{in}\left(x e^{inx} \Big|_0^{\pi} - \int_0^{\pi} e^{inx} dx \right)}$$

$$\cancel{= \frac{1}{in} e^{in\pi} \pi - \frac{1}{(in)^2} e^{inx} \Big|_0^{\pi}}$$

$$\int_{-\pi}^{\pi} t^2 dt = 2\int_0^{\pi} t^2 dt = \frac{2\pi^3}{3}$$

$$\Rightarrow \|f\|^2_{L^2_{2\pi}} = \frac{\pi^2}{3}$$

$$\sum_{n \in \mathbb{Z}} \frac{1}{n^2} = \frac{\pi^2}{3}$$

so $$\sum_{n \in \mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Invertible Operators and Spectrum

<u>Def</u>: $A: \mathfrak{X} \to Y$ $\quad \mathfrak{X}, Y$ Banach, A bdd, linear.

A is <u>left-invertible</u> if $\exists\, B \in \mathcal{B}(Y, \mathfrak{X})$ s.t. $BA = I_{\mathfrak{X}}$.

A is <u>right-invertible</u> if $\exists\, C \in \mathcal{B}(Y, \mathfrak{X})$ s.t. $AC = I_Y$.

A is <u>invertible</u> if it is left and right invertible.

<u>Prop</u>: A is invertible $\Rightarrow$ Left and right inverses are unique and coincide.

<u>Pf</u>: $BA = I \Rightarrow BAC = C \Rightarrow BI = C \Rightarrow B = C$

For uniqueness, fix one of the operators.

<u>Prop</u>: If $\mathfrak{X}, Y$ are Hilbert, then $A \in \mathcal{B}(\mathfrak{X}, Y)$ is invertible iff A^* is invertible.

$$(A^*)^{-1} = (A^{-1})^*$$

<u>Pf</u>: Exercise to the reader

<u>Lemma</u>: $(A^*)^* = A$

Reasons for non-Invertibility

(1) $\operatorname{Ker} A \neq \{0\}$

(2) $\operatorname{Ker} A^* \neq \{0\}$

(3) More reasons in infinite dimensions

Ex: $L^2(0,1)$ $\quad A = M_x \quad A f(x) = x f(x)$

$A = A^* \quad \text{Ker}(A) = \{0\}$

But A is not invertible, because $A^{-1} = M_{1/x}$ is unbounded.

Other Reasons for non-Invertibility

(3) (a) $\exists\ x_n \in \mathcal{X}$, $\|x_n\| = 1$ s.t. $\|Ax_n\| \to 0$

(b) Same for A^*

Relations Between Ranges and Kernels

Thm: $\text{Ker } A = (\text{Range} A^*)^\perp$

$\text{Ker } A^* = (\text{Range} A)^\perp$

$\text{Cl Range } A = (\text{Ker } A^*)^\perp$

$\text{Cl Range } A^* = (\text{Ker } A)^\perp$