Lecture 17

Ker A: $\quad$ ($A : X \to Y$ - Hilbert)

1) $Ax = 0 \Leftrightarrow (Ax, y) = 0 \ \forall y \in Y \Leftrightarrow (x, A^* y) = 0 \Leftrightarrow (x, z) = 0 \quad \forall z \in \operatorname{Ran}$

$\Leftrightarrow x \perp \operatorname{Ran} A^* \Leftrightarrow x \in (\operatorname{Ran} A^*)^\perp$

$\operatorname{Ker} A^* = (\operatorname{Ran} A)^\perp$ follows from 1: b/c $(A^*)^* = A$

Lemma $x \perp E \Leftrightarrow x \perp \operatorname{Clos} E$

$\Leftarrow$ trivial

$\Rightarrow$ take $y \in \operatorname{Clos} E \Rightarrow \exists \ y_n \in E$ s.t. $y_n \to y$

$\|y - y_n\| \to 0$

$\lim (y_n, x) = (y, x)$ because $L_y = (y, x)$ bdd. lin. functional on n goes to infty

$(y_n, x) = 0 \Rightarrow (y, x) = 0$

Cor. E linear manifold (not necessarily closed)

$(E^\perp)^\perp = \operatorname{clos} E$

Proof of prop. $(\operatorname{Ran} A)^\perp = (\operatorname{Clos} \operatorname{Ran} A)^\perp$

$\parallel$

$\operatorname{Ker} A^*$

$(\operatorname{Ker} A^*)^\perp = (\operatorname{Clos} \operatorname{Ran} A)^{\perp\perp} = \operatorname{Clos} \operatorname{Ran} A$

E - closed subspace $\Rightarrow (E^\perp)^\perp = E$

$A = M_x$ on $L^2(0,1)$; $\chi_A(t) = \begin{cases} 1 & t \in A \\ 0 & t \notin A \end{cases}$

$g_n = \chi_{(1/n, 1]} \Rightarrow g_n = A f_n \qquad f_n(x) = \frac{1}{x} g_n(x)$

$f_n \in L^2(0,1)$

$L^2\text{-}\lim_{n \to \infty} g_n = 1 \ (\chi_{(0,1)}) \notin \operatorname{Ran} A$ b/c $f(x) = \frac{1}{x} \notin L^2$

Thm. Banach inverse map thm.

$A \in B(X,Y)$, X, Y - Banach, A - bdd, bij $\left(\operatorname{Ker} A = \{0\}, \operatorname{Ran} A = Y\right)$ Then A - invertible

A^{-1} is bounded

Reasons for non invertibility

1. $\operatorname{Ker} A \neq \{0\}$
2. $\operatorname{Ker} A^* \neq \{0\}$
3. $\operatorname{Ran} A$ - not closed
4. $\operatorname{Ran} A^*$ - not closed

If $\operatorname{Ker} A = \{0\} \Rightarrow \operatorname{Ran} A^*$ is dense
$\operatorname{Ker} A^* = \{0\} \Rightarrow \operatorname{Ran} A$ - dense

Spectrum of operator

$A: X \to X$ - Banach

Def. Resolvent set $\rho(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is invertible}\}$

Def. spectrum $\sigma(A) = \mathbb{C} - \rho(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}$

Ex: 1. $\varphi \in C[0,1]$

$A = M_\varphi$ in $L^2[0,1]$

$Af = \varphi \cdot f$

$\sigma(A) = \varphi([0,1]) = \{y = \varphi(x) : x \in [0,1]\}$

If $\lambda \notin \varphi([0,1]) \Rightarrow \frac{1}{\varphi - \lambda} \in C[0,1]$, so $M_{\frac{1}{\varphi-\lambda}} = (A - \lambda I)^{-1}$, $A - \lambda I = M_{(\varphi - \lambda)}$

If $\lambda \in \varphi([0,1]) \Rightarrow \frac{1}{\varphi - \lambda}$ unbounded, so $M_{\frac{1}{\varphi-\lambda}}$ unbounded

2. Shift operator

$S : \ell^2 \to \ell^2$

$S(x_0, x_1, x_2 \ldots) = (0, x_0, x_1, \ldots)$

$S^*(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots)$

$\sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$

Rmk (Hilbert space case)

Since $(\alpha A)^* = \bar{\alpha} A^*$

$\Rightarrow A - \lambda I$ is inv. $\Leftrightarrow A^* - \bar{\lambda} I$ inv. $\Rightarrow \sigma(A^*) = \overline{\sigma(A)} = \{\bar{\lambda} : \lambda \in \sigma(A)\}$

Thm $A \in B(X)$ X - Banach $\Rightarrow \sigma(A)$ is non-empty compact subset of $\mathbb{C}$ (closed, bounded)

L1 If $\|A\| < 1$ then $I - A$ invertible and $(I-A)^{-1} = \sum_{n=0}^{\infty} A^n = I + A + A^2 + \ldots$ (von Neumann)

Pf $\sum \|A\|^n$ converges $\Rightarrow \sum A^n$ converges

$\|A^n\| \leq \|A\|^n$

$(I-A) \sum_{n=0}^{\infty} A^n = I$

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s.6.3 on Handout

\# 6.16, \# 6.17