10/15/68

Lemma 1

If $\|A\| < 1$ then $I-A$ is invertible & $(I-A)^{-1} = \sum_{n=0}^{\infty} A^n$

Goal: To show $\sigma(A)$ is non-empty + compact.

Cor:

If $|\lambda| > \|A\| \Rightarrow A - \lambda I$ is invertible

Pf $A - \lambda I = -\lambda(I - \frac{A}{\lambda})$ and $\|\frac{A}{\lambda}\| < 1$.

Cor:

$\sigma(A) \subset \{z \in \mathbb{C} : |z| \le \|A\|$.

Lemma 2

If A is invertible & $\|B\| < 1/\|A\|$, then $A-B$ is invertible

and $(A-B)^{-1} = \sum_{n=0}^{\infty} (A^{-1}B)^n A^{-1}$.

Pf:

$A - B = A(I - A^{-1}B)$

$\|A^{-1}B\| \le \|A^{-1}\| \, \|B\| \le 1$

$\Rightarrow (A-B)^{-1} = (I - A^{-1}B)^{-1} A^{-1} = \sum_{n=0}^{\infty} (A^{-1}B)^n A^{-1}$.

Cor: If $A - \lambda_0 I$ is invertible, then $\lambda_0 \notin \sigma(A)$

and $|\lambda - \lambda_0| < \dfrac{1}{\|(A-\lambda_0 I)^{-1}\|} \Rightarrow A - \lambda I$ is invertible.

Cor: $\rho(A)$ is open, ($\sigma(A)$ is closed).

So $\sigma(A)$ is compact.

Why is spectrum non-empty?

Analytic function on $\underset{\text{open}}{\Omega} \subset \mathbb{C}$ is a function s.t. $\forall\, z_0 \in \Omega\ \exists$ r s.t. $D_{z_0} \subset \Omega$ and $f(z) = \sum_0^\infty a_k (z-z_0)^k \quad |z-z_0| < r.$

Cor

$(A - zI)^{-1} \quad z \in \rho(A) = \sigma(A)^c$ is analytic.

$$(A - \lambda I) = ((A-\lambda_0 I) - (\lambda-\lambda_0) I)$$

$$(A-\lambda I)^{-1} = \sum_{n=0}^{\infty} [(A-\lambda_0 I)^{-1}(\lambda-\lambda_0)]^n (A-\lambda_0 I)^{-1}$$

$$= \sum_{n=0}^{\infty} (A-\lambda_0 I)^{-n-1} (\lambda - \lambda_0).$$

If $x, y \in \mathcal{H}$ are fixed, then $((A-zI)^{-1}x, y)$ is analytic on $\rho(A)$.

Lemma If $|\lambda| > 2\|A\|$, then $\|(A-\lambda I)^{-1}\| \le \frac{2}{|\lambda|}$.

Pf

$$A - \lambda I = -\lambda\left(I - \frac{A}{\lambda}\right)$$

$$\|(A-\lambda I)^{-1}\| = \left\| -\frac{1}{\lambda} \sum_{n=0}^{\infty} A^n \lambda^{-n} \right\| \le \frac{1}{\|\lambda\|} \cdot \sum \left\|\frac{A}{\lambda}\right\|^n \le \frac{1}{\|\lambda\|} \cdot \sum \left(\frac{1}{2}\right)^n$$

$$= \frac{2}{|\lambda|}$$

Suppose $\sigma(A) = \emptyset$

So $(A-\lambda I)^{-1}$ is defined $\forall \lambda \in \mathbb{C}$.

$\Rightarrow \exists\ C < \infty \quad \|(A-\lambda I)^{-1}\| \le C$.

Why? If $|\lambda| > 2\|A\|$

then $\|(A-\lambda I)^{-1}\| \le \dfrac{2}{|\lambda|} \le \dfrac{1}{\|A\|}$ $\Rightarrow$ Always bounded.

$(A-\lambda I)^{-1}$ analytic $\Rightarrow$ continuous

so it is bounded on the set $\{\lambda\}$ s.t. $\{|\lambda| \le 2\|A\|\}$

So $\forall x, y \in \mathcal{H}$, $((A-\lambda I)^{-1}x, y) = f(z)$.

We know $f(z)$ is analytic in $\mathbb{C}$.

$|f(z)| \le C\|x\|\|y\|$ bounded. $\leftarrow$

Liouville Thm: If f is analytic + bdd in $\mathbb{C}$, then $f \equiv$ constant.

$\Rightarrow f(z) \equiv$ constant.

$|f(z)| \le \dfrac{2}{|z|}\|x\|\cdot\|y\|$ if $|z| > 2\|A\|$.

$\Rightarrow$ constant $\to 0$.

$\Rightarrow f(z) \equiv 0$.

$\Longrightarrow ((A-zI)^{-1}x, y) = 0 \quad \forall x, y \quad \forall z$

$\Rightarrow (A-zI)^{-1} = 0 \quad \forall z$, but zero cannot be inverse.

$\Longrightarrow \sigma(A)$ is non-empty.

Spectral Radius

$r(A) \equiv \max\{|\lambda| : \lambda \in \sigma(A)\}$

$r(A) \le \|A\|$ (Cor: if $\lambda > \|A\|$, then $A - \lambda I$ is invertible)

Shift

$\sigma(S) = \overline{\sigma(S^*)}$ $\quad (\sigma(A) = \overline{\sigma(A^*)})$

S has no eigenvalues / vectors:

$S^*(x_0, x_1, x_2 \ldots) = (x_1, x_2, x_3 \ldots)$.

Solve $S^* x = \lambda x$.

$\lambda = \dfrac{x_{n+1}}{x_n}$ $\quad x_0(1, \lambda, \lambda^2, \ldots)$ is eigenvector.

must be in ℓ^2 so $|\lambda| < 1$.

We know $\sigma(S^*) \supset \{z \in \mathbb{C} : |z| < 1\}$

$\Rightarrow \sigma(S^*) \supset \{z \in \mathbb{C} : |z| \le 1\}$ since we know $\sigma(A)$ is closed

$\rho(S^*) \le \|S^*\| = \|S\| = 1$.

$\Rightarrow \sigma(S^*) = \{z \in \mathbb{C} : |z| \le 1\}$