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Spectral Mapping Theorem

$$p(z) = \sum_{k=0}^{n} a_k z^k \qquad p(A) = \sum_{k=0}^{n} a_k A^k$$

$$A^j A^k = A^{j+k}$$

<u>Thm</u>: $\sigma(p(A)) = p(\sigma(A)) = \{ p(\lambda) : \lambda \in \sigma(A) \}$

<u>Pf</u>: $g(z) = p(z) - \mu \quad \mu \in \mathbb{C}$, by the fundamental theorem of algebra

$$g(z) = p(z) - \mu = a(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n)$$

$\mu \notin \sigma(p(A)) \iff p(A) - \mu$ is invertible $\iff a(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_n I)$ is invertible

$\overset{*}{\iff} (A - \lambda_k I)$ invertible

$\iff$ all $\lambda_k \notin \sigma(A)$

$\iff \mu \notin p(\sigma(A))$

<u>Pf of *</u>: ($\Leftarrow$) trivial

($\Rightarrow$) <u>Lemma</u>: If AB is invertible, then A is right invertible and B is left invertible.

<u>Pf</u>: $A \underbrace{B(AB)^{-1}}_{\text{right inv.}} = I$

$\underbrace{(AB)^{-1}A}_{\text{left inv.}} B = I$

Since all $(A - \lambda_k I)$ commute, applying lemma gives right and left invertibility of each $(A - \lambda_k I)$ $\square$

<u>Ex of Lemma</u>: $S^* S = I$ but S, S^* are not invertible

<u>Cor</u>: $T^n = 0 \Rightarrow \sigma(T) = \{0\}$

Def: If $T^n = 0$ then T is nilpotent.

Gelfand's Formula for $r(A)$

Thm: $r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$

Pf: ($\le$) $r(A) \le \|A\| \Rightarrow r(A)^n \underset{\substack{\uparrow \\ \text{spectral} \\ \text{mapping}}}{=} r(A^n) \le \|A^n\|$

$\Rightarrow r(A) \le \|A^n\|^{1/n}$

($\ge$) $x_n \in \mathfrak{X}$ Banach, $\sum x_n$ converges $\Rightarrow \|x_n\| \to 0$ as $n\to\infty$

$$\underbrace{(A-\lambda I)^{-1}}_{\substack{\text{defined for} \\ |\lambda|>r(A)}} = -\frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{A^n}{\lambda^n} \qquad |\lambda| > \|A\|$$

choose $\alpha > 0$ s.t. $|\lambda| = r(A) + \alpha$

then $\dfrac{\|A^n\|}{(r(A)+\alpha)^n} \longrightarrow 0$

$\Rightarrow \exists N$ s.t. $\forall n > N$ $\dfrac{\|A^n\|}{(r(A)+\alpha)^n} < 1$

$\Rightarrow \|A^n\|^{1/n} < r(A) + \alpha$ $\qquad$ α was arbitrary

$\Rightarrow \|A^n\|^{1/n} \le r(A)$

Cor: If $\lim \|A^n\|^{1/n} = 0$ then $\sigma(A) = \{0\}$

Def: If $\sigma(A) = \{0\}$ then A is quasinilpotent.

<u>Thm</u>: If $N^*N = NN^*$, then $r(N) = \|N\|$

<u>Lemma</u>: $\|A^*A\| = \|A\|^2$

<u>Pf</u>: $(\leq)$ $\|A^*A\| \leq \|A^*\| \|A\| = \|A\|^2$

$(\geq)$ $$\|A\|^2 = \sup_{\|x\| \leq 1} \|Ax\|^2$$

$$= \sup_{\|x\| \leq 1} (Ax, Ax)$$

$$= \sup (A^*Ax, x) \leq \|A^*A\|$$

<u>Lemma</u>: $A = A^* \Rightarrow \|A\| = r(A)$

<u>Pf</u>: $\|A^2\| = \|A^*A\| = \|A\|^2$

$$\Rightarrow \|A^{2^n}\| = \|A\|^{2^n}$$

$$\Rightarrow \lim_{n \to \infty} \|A^n\|^{1/n} = \|A\|$$

<u>Cor</u>: $A = A^*$ then $\|A^n\| = r(A^n) = r(A)^n = \|A\|^n$

<u>Pf of Thm</u>: $\|N^n\|^2 = \|N^{*n} N^n\| = \|(N^*N)^n\| = \|N^*N\|^n = \|N\|^{2n}$

$$\Rightarrow \|N^n\|^{1/n} = \|N\|$$

<u>Ex</u>: S^* has representation $$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & 0 \\ & & \ddots & \ddots & \\ & 0 & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}$$

$$\sigma(S^*) = \mathrm{cl}\, \mathbb{D}$$

Consider $T = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$ $\quad T^n = 0 \quad \Rightarrow \sigma(T_n) = \{0\}$

So going from finite to infinite has significant impact on spectrum

Ex: $A : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$

$$A\{x_k\}_{k\in\mathbb{Z}} = \{x_{k+1}\}$$

$$\sigma(A) = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$