# Lecture 20

$$A\{x_n\}_{n=-\infty}^{\infty} = \{x_{n+1}\}_{n=-\infty}^{\infty}$$

$\ell^2$ $\quad \dots \ x_{-2} \ \ x_{-1} \ x_0 \ \ x_1 \dots$

$\qquad \dots \ x_{-1} \ \ x_0 \ \ x_1 \ \ x_2 \dots$

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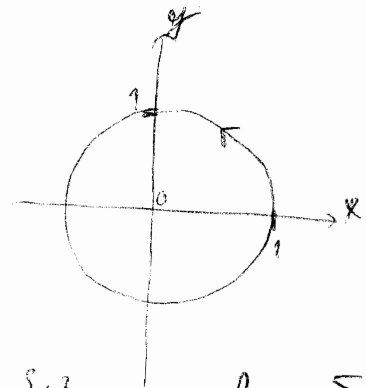
$$\{x_n\}_{n\in\mathbb{Z}} \longleftrightarrow \sum_{n\in\mathbb{Z}} x_n e^{int} = f(t) \in L^2_{2\pi}$$

$\{x_n\} \in \ell^2 \qquad e^{it} = z$

$$\int_0^{2\pi} |f(e^{it})|^2 \frac{dt}{2\pi} = \sum |x_n|^2$$



$$\{x_n\}_{n\in\mathbb{Z}} \longleftrightarrow \underbrace{f(z) = \sum_{n\in\mathbb{Z}} x_n z^n}_{L^2(\mathbb{T})}$$

<u>Claim</u> $\quad Af(z) = \bar{z} f(z) = z^{-1} f(z)$

$\qquad \sigma(A) = \{z^{-1} : z \in \mathbb{T}\} = \{e^{-it} : t \in \mathbb{R}\} = \mathbb{T}$

## Gelfand formula

$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$ works on $A: X \to X$, $X$ - Banach

<u>Example</u> Volterra integral operator

$\qquad C[0,a]$, $K(x,y)$, $K \in C([0,a]\times[0,a])$

$$Tf(x) = \int_0^x K(x,y) f(y)\, dy$$

<u>Claim.</u> $r(T) = 0 \Rightarrow \sigma(T) = \{0\}$

$\|T^n\|$ $\qquad$ $K$ continuous on $[0,a]\times[0,a]$ so $\exists\ c < \infty$ s.t. $|K(x,y)| \le C$

- let $f \in C[0,1]$, $\|f\| \le 1$
- let's estimate $T^n f$

$$|Tf(x)| \le \int_0^x |K(x,y) f(y)|\, dy \le \int_0^x C \cdot 1\, dy = Cx$$

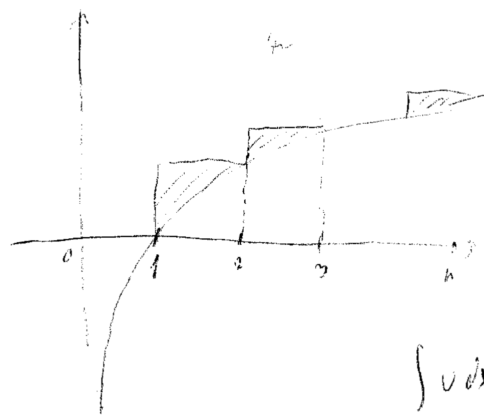
$$|T^2 f(x)| \le \left|\int_0^x K(x,y)\, Tf(y)\right| dy \le \int_0^x C \cdot Cy\, dy \le \frac{(Cx)^2}{2}$$

$$|T^3 f(x)| \le \frac{(Cx)^3}{3!}$$

So $|T^n f(x)| \le \dfrac{(Cx)^n}{n!} \Rightarrow \|T^n f\| \le \dfrac{(Ca)^n}{n!}$, $\|T^n\|^{\frac{1}{n}} \le \dfrac{Ca}{(n!)^{\frac{1}{n}}} \longrightarrow 0$

Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \iff \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} \xrightarrow{n\to\infty} 1$$




$$\int u\,dv = uv - \int v\,du$$

$$\ln n! = \ln 1 + \ln 2 + \dots + \ln n \geq \int_1^n \underbrace{\ln x}_{u}\,\underbrace{dx}_{dv} =$$

$$= x\ln x\Big|_1^n - \int_1^n x\, d\ln x$$

$$= n\ln n - (n-1)$$

$$\frac{\ln n!}{n} \geq \ln n - \frac{n-1}{n} \geq \ln n - 1$$

$$(n!)^{\frac{1}{n}} \geq \frac{n}{e}$$

$$\frac{Ca}{(n!)^{\frac{1}{n}}} \leq \frac{Cae}{n} \xrightarrow[n\to\infty]{} 0$$

Applications to DE

$$\begin{cases} y'(x) = p(x)y(x) + g(x) \\ y(0) = y_0 \end{cases}$$

$$y(x) = y_0 + \int_0^x y'(t)\,dt = y_0 + Q(x) + \int_0^x p(t)y(t)\,dt$$

$$Q(x) = \int_0^x g(t)\,dt$$

$[0,a]$

$$y - Ty = y_0 + Q$$

$$[Ty](x) = \int_0^x p(t)y(t)\,dt$$

$$\sigma(T) = 0$$

$$\underbrace{(I-T)}_{\text{invertible}}y = \underbrace{y_0+Q}_{f}$$

Remark $f, K$ are not scalar valued, but vector valued

$f(x) \in \mathbb{R}^n$

$K(x,y) \in M_{n\times n}$

Functions of self-adjoint operators

$A^* = A$ , $\sigma(A) \subset [a,b]$

$f(A)$, $f \in C[a,b]$

$\sqrt{A}$ if $\sigma(A) \subset [0,\infty)$

Weierstrass thm. polynomials are dense in $C[a,b]$

$\left.\begin{array}{l} p_n \rightrightarrows f \\ \lim p_n(A) \end{array}\right\}$ Ex (Cauchy seq.)

L $\|p(A)\| = \underbrace{\max_{x \in \sigma(A)} |p(x)|}$
$A = A^*$ $\quad$ $\| \leftarrow$ by spectral mapping thm $\left(\sigma(p(A)) = p(\sigma(A))\right)$

$\sigma(p(a))$

$p(x) = \sum_0^m a_k x^k \quad a_k \in \mathbb{R}$

$A = A^* \Rightarrow p(A)^* = p(A)$