10/22

Was last time

Lemma: $\|p(A)\| = \sup_{x\in\sigma(A)} |p(x)| \le \|p\|_{C[a,b]}$ where $A = A^*$.

Prop: If $p_n \to f$ in $C[a,b]$ then $p_n(A)$ converges.

Pf: p_n converges $\Rightarrow$ Cauchy, so $\forall\ \varepsilon > 0\ \exists N$ s.t. $\forall m,n \ge N$

$\|p_n - p_m\|_{C[a,b]} < \varepsilon$.

But $\|p_n(A) - p_m(A)\| = \|(p_n - p_m)(A)\| \le \|p_n - p_m\|_{C[a,b]} < \varepsilon$.

So $p_n(A)$ is Cauchy so converges, and we call the limit $f(A)$.

Well-defined - $p_n \to f$, $q_n \to f$ in $C[a,b]$, then

$\|p_n(A) - q_n(A)\| \le \|p_n - q_n\|_{C[a,b]} \to 0$ as $n \to \infty$

so $\lim p_n(A) = \lim q_n(A)$.

If $\sigma(A) \subset [0,\infty)$ one can define $\sqrt{A}$ with $(\sqrt{A})^2 = A$

Cor: $A = A^*$, $\sigma(A) \subset [0,\infty)$, then $\|A\| = \sup_{\|x\|\le 1} (Ax, x)$

Pf: Exercise

Remark: $A = A^*$ then $\|A\| = \sup_{\|x\|\le 1} |(Ax,x)|$

Weierstrass Theorem: $f \in C[a,b] \Rightarrow \exists\, p_n \rightrightarrows f$ on $[a,b]$.

Mollifiers: take φ s.t. $\int_{-\infty}^{\infty} \varphi(x)\,dx = 1$, $\int_{-\infty}^{\infty} |\varphi(x)|\,dx < \infty$

Given $\alpha > 0$, define $\varphi_\alpha(x) = \frac{1}{\alpha}\varphi(\frac{x}{\alpha})$

then $\int_{-\infty}^{\infty} \varphi(x)\,dx = \int_{-\infty}^{\infty} \varphi_\alpha(x)\,dx$ by change of variables

Lemma: f is bounded and uniformly continuous on $\mathbb{R}$

$$f_\alpha(s) := \int_{-\infty}^{\infty} f(t)\, \varphi_\alpha(s-t)\, dt = f * \varphi_\alpha .$$

Then $f_\alpha \rightrightarrows f$ as $\alpha \to 0$

Pf:
$$|f(s) - f_\alpha(s)| = \left| \int_{\mathbb{R}} f(s-t)\,\varphi_\alpha(t)\,dt - f(s) \right|$$
$$= \left| \int_{\mathbb{R}} [\, f(s-t) - f(s) \,]\, \varphi_\alpha(t)\, dt \right| \le \left| \int_{-\delta}^{+\delta} \right| + \left| \int_{rest} \right|$$

Now $\varepsilon > 0$, f is uniformly continuous, so $\exists\, \delta$ s.t. $\forall\, s,t$

$|t| < \delta \Rightarrow |f(s) - f(s-t)| < \frac{\varepsilon}{3C}$ where $|f(s)| < M$, $\int_{\mathbb{R}} |\varphi|\, dx = \int_{\mathbb{R}} |\varphi_\alpha|\, dx < C$

$$\left| \int_{-\delta}^{\delta} (f(s-t) - f(s))\, \varphi_\alpha(t)\, dt \right| \le \int_{-\delta}^{\delta} \frac{\varepsilon}{2C} |\varphi_\alpha(t)|\, dt \le \int_{-\infty}^{\infty} \frac{\varepsilon}{2C} |\varphi_\alpha(t)|\, dt \le \frac{\varepsilon}{3} < \frac{\varepsilon}{2}$$

$$\left| \int_{\mathbb{R} \setminus [-\delta, \delta]} \varphi_\alpha(t) [\, f(s-t) - f(s) \,]\, dt \right| \le \int_{\mathbb{R} \setminus [-\delta,\delta]} |\varphi_\alpha(t)|\, 2M\, dt = 2M \int_{\mathbb{R} \setminus [-\frac{\delta}{\alpha}, \frac{\delta}{\alpha}]} |\varphi(x)|\, dx \quad \text{where } x = \frac{t}{\alpha}$$

as $\alpha \to 0$, $-\frac{\delta}{\alpha} \to -\infty$, $\frac{\delta}{\alpha} \to \infty$

$\Rightarrow \int |\varphi(x)|\, dx \to 0$ as $\alpha \to 0$

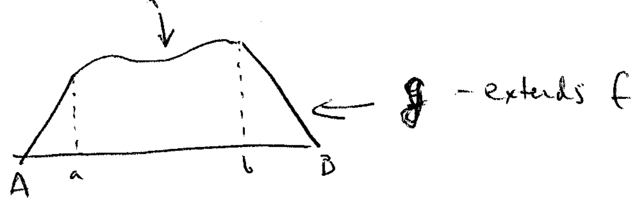
$\Rightarrow \int_{rest} < \frac{\varepsilon}{2} \;\; \forall\, \alpha < \alpha_0.$

$\Rightarrow |\varphi_\alpha * f(s) - f(s)| < \varepsilon$

Any function with compact support is bounded and uniformly continuous.

Remark: The lemma is true for L^p convergence with $1 \le p < \infty$.

Pf of WT:



$\varphi(x) = ae^{-x^2}$ $\qquad g_\varepsilon(s) = g * \varphi_\varepsilon(s)$

$g_\alpha(s) = \int g(t)\, \varphi_\alpha(s-t)\, dt$

$g_\alpha \rightrightarrows g$ as $\alpha \to 0$

by the previous lemma