Weierstrass Theorem Proof (continued)

$$\int g(t)(s-t)^n \, dt = \int g(t) \cdot \sum_{k=0}^{\infty} \binom{n}{k} s^{n-k} (-t)^k \, dt.$$

$$= \left[\sum_{k=0}^{n} \binom{n}{k} \int g(t)(-t)^k \, dt \right] s^{n-k}$$

$g \in C_0$ $\qquad g * \text{Poly} = \text{Polynomial}.$

$\phi = a e^{-x^2}$ $\qquad g_\alpha = g * \Phi_\alpha$ $\qquad \Phi_\alpha(x) = \frac{1}{\alpha} \Phi\left(\frac{x}{\alpha}\right).$

$\forall \varepsilon > 0 \; \exists \alpha$ s.t. $|g(s) - g * \Phi_\alpha(s)| \le \frac{\varepsilon}{3} \; \forall s.$

$g * \Phi_\alpha(s) = \int_A^B g(t) \Phi_\alpha(s-t) \, dt.$ $\qquad$ Claim: can be $\approx$ by poly.

Let $R = B - A$ $\qquad e^{-x^2} = \sum \frac{(-x^2)^n}{n!} \Rightarrow r_{converg} \to \infty$

Taylor series for Φ_α converges uniformly on any bounded interval.

Let $|P_n(x) - \Phi_\alpha(x)| \le \frac{\varepsilon}{3 \|g\| R}$

$$|g * \Phi_\alpha - g * P_n(s)| = \left| \int_A^B g(t) \left[\Phi_\alpha(s-t) - P_n(s-t) \right] dt \right| \quad s \in [A, B].$$

$$\le \int_A^B \|g\| \frac{\varepsilon}{3 \|g\| R} \, dt = \frac{\varepsilon}{3}$$

$\Longrightarrow |g(s) - g * \Phi_\alpha(s)| \le \frac{\varepsilon}{2}$ $\quad \Rightarrow \; |g(s) - \underbrace{(g * P_n)(s)}_{\text{poly.}}| \le \frac{2\varepsilon}{3} < \varepsilon$

$|g * \Phi_\alpha - g * P_n(s)| \le \frac{\varepsilon}{3}$

$s \in [A, B]$ $\quad g(s) = f(s)$ on $[a, b] \subset [A, B]$

Mollifying

$$\frac{d}{ds}[f * g(s)] = \frac{d}{ds}\int_{\mathbb{R}} f(t)\,g(s-t)\,dt. = \lim_{\Delta s\to 0}\int_{\mathbb{R}} \frac{f(t)\,g(s+\Delta s - t) - g(s-t)}{\Delta s}\,dt.$$

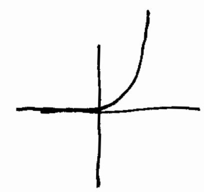
$$= \int_{\mathbb{R}} f(t)\,g'(s-t)\,dt.$$

If $g \in C_0^\infty$ → inf. diff. → compact support.

You can interchang differentiation and integral.

C^∞ glue

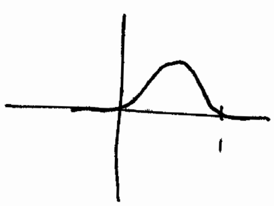
$$\phi^1(x) = \begin{cases} 0 & x \le 0 \\ e^{-\frac{1}{x^2}} & x > 0. \end{cases} \qquad \frac{d}{dx}\left(\Phi^1(x)\right)\Big|_{x=0} = 0.$$



$$0 \le \theta(x) \le 1$$

$$\phi'(0) = \lim_{x\to 0}\frac{\phi(x)-\phi(0)}{x} = \lim_{x\to 0}\phi'(\theta(x)\cdot x) = \lim_{y\to 0}\phi'(y) = 0.$$

$$\phi^2(x) = \phi^1(x)\cdot\phi^1(1-x)$$

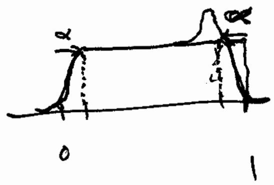


$$\Phi(x) = \Phi^2(x) \Big/ \int \Phi^2(x)\,dx. \qquad \int \Phi(x)\,dx = 1, \quad \Phi \in C_0^\infty$$

$f \in$ Bdd Unif. Cont.

$\Rightarrow$ ~~f(x)~~ $f * \Phi_\alpha \rightrightarrows f.$

$\chi[0,1] * \Phi_\alpha$



C_0^∞

Corollary: C_0^∞ functions are dense in $L^p(\mathbb{R})$ $1 \le p < \infty$

Pf

C_0^∞ is dense in C_0

C_0 is dense in L_0^p.

L_0^p is dense in L^p. ∎

If $f \in L^p$, $f_n = f\chi[-n,n]$, $\int |f-f_n|^p dt = \int |f|^p dt - \int_{-n}^{n} |f|^p dt$.

$\longrightarrow 0.$