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Fourier Transform

$f \in L^2(-\pi,\pi)$, then $f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikx}$

$$\text{where } \hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-ikt}\, dt$$

$f \in L^2(-\pi R, \pi R)$ $\quad x = Ru \quad u \in (-\pi,\pi) \quad g(u) = f(x) = f(Ru)$

$$f(x) = g(u) = \sum_{k \in \mathbb{Z}} \hat{g}(k) e^{iku}$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(Ru) e^{-iku}\, du \right) e^{ik\frac{x}{R}}$$

define $t = Ru$

$$= \sum_{k \in \mathbb{Z}} \underbrace{\left(\frac{1}{2\pi R} \int_{-\pi R}^{\pi R} f(t) e^{-ik\frac{t}{R}}\, dt \right)}_{\hat{f}(k)} e^{ik\frac{x}{R}}$$

$$= \sum_{k \in \mathbb{Z}} \left(\frac{1}{\sqrt{2\pi}} \int_{-\pi R}^{\pi R} f(t) e^{-ik\frac{t}{R}}\, dt \right) \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{R} e^{ik\frac{x}{R}}$$

Assume $f \in C_0^\infty$ then we can integrate over all of $\mathbb{R}$

Define $\xi = \frac{k}{R}$

$$\Rightarrow \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-it\xi}\, dt$$

$$f(x) \simeq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x}\, d\xi$$

Riemann sum for this $\int$ with step $\frac{1}{R}$

These are in fact equal because the limit as $R \to \infty$ is the integral

Def: $\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx = \mathcal{F} f \qquad f \in L^1$

$$\mathcal{F}^{-1} g = \check{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\xi) e^{i\xi x}\, d\xi \qquad g \in L^1$$

What is $\mathcal{F}^*$?

$$(\mathcal{F}f, g) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx\right)\overline{g(\xi)}\,d\xi$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\Bigg(\underbrace{\overline{\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\xi x} g(\xi)\,d\xi}}_{\mathcal{F}^{-1}}\Bigg)f(x)\,dx \quad = (f, \mathcal{F}^{-1}g)$$

for $f, g \in C_0^\infty$ we can change the order of integration $\Rightarrow \mathcal{F}^{-1} = \mathcal{F}^*$

<u>Claim</u>: $\mathcal{F}$ is unitary $(\mathcal{F}^{-1} = \mathcal{F}^*)$

Sufficient to show $\mathcal{F}^*\mathcal{F} = I$, $\mathcal{F}\mathcal{F}^* = I$ follows because $\check{f}(x) = \hat{f}(-x)$

<u>Properties of Fourier Transform</u>:

① If $f \in C_0^\infty$ then $\forall\, n>0$ $\exists\, C = C(n,f)$ s.t.

$$|\hat{f}(\xi)| \le \frac{C}{|\xi|^n} \qquad |\xi| \ge 1$$

<u>Pf</u>: $\sqrt{2\pi}\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx = \int_{-\infty}^{\infty} f(x)\,d(e^{-i\xi x})\cdot\frac{1}{-i\xi}$

$$= \frac{i}{\xi}\left(\cancel{f(x)e^{-i\xi x}\Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} f'(x)e^{-i\xi x}dx\right)$$

$$= \frac{-i}{\xi}\,\hat{f'}(\xi)\cdot\sqrt{2\pi}$$

$$\Rightarrow \hat{f}(\xi) = \left(\frac{-i}{\xi}\right)^n \widehat{f^{(n)}}(\xi) \qquad \text{Let } C = \|f^{(n)}\|_1$$

$$\Rightarrow \hat{f}(\xi) \le \frac{C}{|\xi|^n} = \frac{\|f\|_{L^1}}{|\xi|^n}$$

② $\widehat{f'}(\xi) = i\xi \hat{f}(\xi)$

③ $\widehat{f*g}(\xi) = \frac{1}{\sqrt{2\pi}} \underbrace{\int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(t)g(s-t)\,dt\right)}_{f*g(s)} \underbrace{e^{-i\xi s}}_{= e^{-i\xi(s-t)} e^{-i\xi t}} ds$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-i\xi t} dt \int_{\mathbb{R}} g(s-t) e^{-i\xi(s-t)} d(s\cancel{\ })$$

$$= \hat{f}(\xi)\,\hat{g}(\xi) \cdot \sqrt{2\pi}$$