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Fourier Inversion Theorem

$$f(x) = \sum_{k\in\mathbb{Z}} \frac{1}{\sqrt{R}} \hat{f}\left(\frac{k}{R}\right) \frac{1}{\sqrt{2\pi R}} e^{i\frac{kx}{R}}$$

$$\mathcal{F}^{-1}\mathcal{F} = \mathcal{F}\mathcal{F}^{-1} = I.$$

If $f \in C_0^\infty$ $\quad \operatorname{supp} f \subset [-\pi R, \pi R]$ $\quad \to \frac{1}{\sqrt{2\pi R}} e^{i\frac{k}{R}x}$

$$f(x) = \sum \underbrace{\frac{1}{\sqrt{2\pi}} \hat{f}\left(\frac{k}{R}\right)}_{\frac{1}{\sqrt{R}}} e^{ik\frac{x}{R}} \cdot \frac{1}{\sqrt{R}}$$

orthonormal in $L^2(-\pi R, \pi R)$

$$\|f\|^2_{L^2(\mathbb{R})} = \|f\|^2_{L^2(-\pi R,\pi R)} = \sum_{k\in\mathbb{Z}} \left|\hat{f}\left(\frac{k}{R}\right)\right|^2 \cdot \frac{1}{R}$$

– Riemann sum for: $\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi$

$\hat{f} \in C^\infty$ and so continuous

Theorem: If f is continuous on $[a,b]$, then f is Riemann integrable ie $\Sigma \to \int$.

$$|\hat{f}(\xi)| \le \frac{\|f^{(n)}\|_{L^1}}{|\xi|^n} \qquad \text{Let } n=2 \Rightarrow \quad |\hat{f}(\xi)| \le \frac{C}{|\xi|^2} \quad |\xi| \ge 1.$$

So given $\varepsilon > 0$, $\exists\, M$ s.t. $\displaystyle\int_{|\xi|\ge M} |\hat{f}(\xi)|^2 d\xi \le \frac{\varepsilon}{4}$

$$\sum_{|\frac{k}{R}|\ge M} \frac{|\hat{f}(\frac{k}{R})|^2}{R} \le \frac{\varepsilon}{4}.$$ and should hold for $\forall R$

Since $\displaystyle\int_{|\xi|\ge M} |f(\xi)|^2 d\xi \le \int_{|\xi|\ge M} \frac{C^2}{|\xi|^4} d\xi \xrightarrow[M\to\infty]{} 0$

$$\Sigma \le \int_{|\xi|\ge M - 1/R}$$

$$\sum_{k\in\mathbb{Z}} \left|\hat{f}\left(\tfrac{k}{R}\right)\right|^2 \frac{1}{R} = \underbrace{\sum_{|\frac{k}{R}|\le M}}_{M\downarrow\ R\to\infty} + \sum_{|\frac{k}{R}|\ge M}$$

$$\| \qquad\qquad \int_{-M}^{M} |\hat{f}(\xi)|^2 d\xi$$

$$\int_{-\infty}^{\infty} \|f\|^2 dx$$

$$\Rightarrow \exists\, R_0 \ \forall\, R > R_0 \text{ s.t. } \left| \sum_{k/R \le M} - \int_{-M}^{M} \right| \le \frac{\varepsilon}{4}$$

$$\left| \int |f|^2 dx - \int |\hat{f}(\xi)|^2 d\xi \right| \le \left| \sum_{|\frac{k}{R}|<M} |\hat{f}(\tfrac{k}{R})|^2 \tfrac{1}{R} - \int_{-M}^{M} |f(\xi)|^2 d\xi \right| + \sum_{|\frac{k}{R}|\ge M} + \int_{|\xi|\ge M} |\hat{f}(\xi)|^2 d\xi$$

$$\le \frac{3}{4}\varepsilon < \varepsilon \quad \forall \varepsilon > 0$$

$$\Rightarrow \int |f|^2 dx = \int |\hat{f}(\xi)|^2 d\xi$$

So $\mathcal{F}$ is isometry $\Longrightarrow$ Unitary. $\mathcal{F}^{-1}$ is isometry.

Properties of Fourier Transform.

$$\widehat{f^{(n)}} = (i\xi)^n \hat{f}(\xi)$$

$$\widehat{(-i\xi)^n f(\xi)} = \left(\frac{d}{d\xi}\right)^n \hat{f}(\xi)$$

Fourier Transformation of Gaussian

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx = 1.$$

$$G(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\hat{G}(\xi) = \frac{1}{\sqrt{2\pi}} \int \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cancel{\xi} e^{-i\xi x} dx. \qquad i\xi = \eta$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2} - \eta x} dx. \qquad \frac{x^2}{2} + \eta x = \frac{1}{2}(x+\eta)^2 - \frac{\eta^2}{2}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+\eta)^2 + \eta^2/2} dx = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}\eta^2} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+\eta)^2} dx}_{=1}$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}\eta^2} = G(\xi)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}}$$

$\hat{G}(\zeta)$ defined $\forall\, \zeta \in \mathbb{C}$.

$\hat{G}(\zeta)$ is analytic in $\mathbb{C}$.

$\frac{1}{\sqrt{2\pi}} e^{-\zeta^2/2}$ is also analytic in $\mathbb{C}$.

We showed that $\hat{G}(\zeta) = \frac{1}{\sqrt{2\pi}} e^{-\zeta^2/2}$ for $i\zeta \in \mathbb{R}$
$\zeta \in i\mathbb{R}$.

<u>Shannon- Kofelnikaer Sampling Theorem</u>

Let $f \in L^2(\mathbb{R})$ $\operatorname{supp} \hat{f} \subset [-\pi R_0, \pi R_0]$

Then $f(\frac{k}{R})$ $k \in \mathbb{Z}$ define f.
for $R \geq R_0$

<u>Pf</u>

$\operatorname{supp} f \subset [-\pi R_0, \pi R_0]$
$\subset [-\pi R, \pi R]$ and $f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(\frac{k}{R}) \frac{1}{\sqrt{2\pi R}} e^{i\frac{k}{R}x}$.

$\frac{1}{\sqrt{2\pi R}}$