Lecture 25

Another way of proving Sh-K Thm.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\pi R}^{\pi R} \hat{f}(\xi) e^{i\xi x} d\xi \quad \text{treat as} \quad g \in L^2(-\pi R, \pi R)$$

$f\left(\frac{k}{R}\right)$ is Fourier coeff # $-k$ up to a constant factor of g

$$\frac{1}{2\pi R} \int_{-\pi R}^{\pi R} \hat{f}(\xi) e^{-\frac{i\xi k}{R}} d\xi = f\left(\frac{k}{R}\right)$$

Fourier coeff. det function on $L^2(-\pi R, \pi R)$

Meaning of Sh-K.

$\cos \omega t \qquad \sin \omega t \qquad e^{i\omega t} \qquad e^{-i\omega t}$

Frequency $= \frac{\omega}{2\pi}$

$\operatorname{supp} \hat{f} \subset [-\pi R_0, \pi R_0] \rightsquigarrow \text{freq} \le \frac{R_0}{2}$

Nyquist critical sampling ~~interval~~ $= \frac{1}{R} = \frac{1}{2\,\text{freq}}$

Caution the setting of Sh-K Thm. is very idealized.

If $\hat{f} \in L^2$, $\operatorname{supp} \hat{f} \in [-\pi R_0, \pi R_0] \Rightarrow f$ is analytic in $\mathbb{C}$

In particular, f cannot have compact supp on $\mathbb{R}$ (in real life we don't have those)

Fourier transform in L^2

$$\frac{1}{\sqrt{2\pi}} \int f(x) e^{-i\xi x} dx \qquad x \in \mathbb{R}$$

, defined for $f \in L^1$ but not for $f \in L^2$

$$\hat{f}(\xi) = \lim_{R\to\infty} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-R}^{R} f(x) e^{-i\xi x} dx}_{\cancel{f(R)}\ f_R \text{, Then take lim in } L^2} , \quad L^2\text{-lim}$$

; $\hat{f} \in L^p$, $p > 1$

What about lim a.e. (almost everywhere)?
Yes L. Carleson 60's for L^2.
Hunt, for $\hat{f} \in L^p$, $p > 1$
(both are very hard theorems)

Poisson summation formula

If $g \in L^1$, ~~$\hat{g} \in$~~ $\hat{g} \in L^1$, then $\sum g(2\pi k) = \frac{1}{\sqrt{2\pi}} \sum_m \hat{g}(m)$

Pf. def $h(x) = \sum_{k \in \mathbb{Z}} g(x + 2\pi k) \Rightarrow h(0) = \sum g(2\pi k)$

$$h(x) = \sum c_m e^{imx} \quad ; \quad c_m = \underbrace{\frac{1}{2\pi}\int_0^{2\pi} h(x) e^{-imx} dx}_{} = \frac{1}{2\pi}\int_{\mathbb{R}} g(x) e^{imx} dx$$

$$\int_0^{2\pi} \sum_k g(x + 2\pi k) e^{-imx} dx$$

$$\sum_{k \in \mathbb{Z}} \int_{2\pi k}^{2\pi(k+1)} g(x)\, dx = \frac{1}{\sqrt{2\pi}} \hat{g}(m)$$

$$h(0) = \sum c_m = \frac{1}{\sqrt{2\pi}} \sum \hat{g}(m)$$

Need $g' \in L^1$
for $h' \in L^1(0, 2\pi)$
so h satisfies Dini Condition
so Fourier series for h
converges pointwise