11/3

$$i\psi_t = \Delta\psi + H\psi \qquad \psi(x,t)$$

$$\int |\psi(x,t)|^2\,dx = \text{constant} = 1$$

$$\int |\hat\psi|^2\,d\xi = 1$$

$|\psi|^2$ - coordinate $\qquad |\hat\psi|$ - momentum

$$\text{Prob}(\text{coord} \in [a,b]) = \int_a^b |\psi|^2\,dx$$

x_m = Mean value $= \int x|\psi|^2\,dx \qquad$ MV of coordinate

$\xi_m = \int \xi |\hat\psi|^2\,d\xi \qquad$ MV of momentum

Dispersion of $x = \mathbb{E}|(x-x_m)|^2 = \int (x-x_m)^2 |\psi(x)|^2\,dx$

$$D\xi = \int (\xi-\xi_m)^2 |\hat\psi(\xi)|^2\,d\xi$$

<u>Thm</u>: $\displaystyle\int_{\mathbb{R}} (x-x_m)^2|\psi(x)|^2\,dx \int_{\mathbb{R}} (\xi-\xi_m)^2|\hat\psi(\xi)|^2\,d\xi \geq \frac{1}{4}$

<u>Pf</u>: begin with assumption $x_m = \xi_m = 0$

Then we must show $\displaystyle\int |x\psi(x)|^2\,dx \underbrace{\int |\xi\hat\psi(\xi)|^2\,d\xi}_{\int |\psi'(x)|^2\,dx} \geq \frac{1}{4}$

Cauchy-Schwarz

$$\Rightarrow \int |x\psi(x)|^2 dx \int |\xi\hat\psi(\xi)|^2 d\xi \geq \left(\int |x\psi(x)\psi'(x)|^2\,dx\right)^2$$

$$\int_{\mathbb{R}} |x\psi(x)\psi'(x)|\,dx \geq \left|\int \text{Re}(x f(x)\overline{f'(x)})\,dx\right|$$

$$= \frac{1}{2}\left|\int x\left(f(x)\overline{f'(x)} + \bar f(x) f'(x)\right)dx\right|$$

$$= \frac{1}{2}\left|\int_{\mathbb{R}} \underset{u}{x}\, \underbrace{\frac{d}{dx}|f(x)|^2\,dx}_{dv}\right|$$

$$= \frac{1}{2}\left|\underbrace{x|\psi(x)|^2\Big|_{-\infty}^{\infty}}_{0 \text{ for } \psi \in C_0^\infty} - \underbrace{\int_{-\infty}^{\infty} |\psi(x)|^2\,dx}_{1}\right| = \frac{1}{2}$$

For the general case $\Psi(x+x_m)\, e^{i q_m x} = f(x)$ has medium value of $x=0$

Discrete Fourier Transform

__Def__: __C_n__ is the cyclic group $\mathbb{Z}/n\mathbb{Z}$

A natural representation of n^{th} roots of unity $e^{\frac{2\pi i k}{n}}$ $\quad 0 \le k < n$

$$\ell^2(C_N) \cong \left\{ \{x_k\}_{k=0}^{N-1},\ \|x\| = \left(\sum_{k=0}^{N-1} |x_k|^2 \right)^{1/2} \right\}$$

$$e_k(n) := \exp\left(\frac{i k n 2\pi}{N} \right) \qquad 0 \le n < N \qquad 0 \le k < N$$

$\{e_k\}_0^{N-1}$ is an orthogonal system, in fact a basis in $\ell^2(C_N)$

taking $\left\{ \frac{e_k}{\sqrt{N}} \right\}$ gives an orthonormal basis

__Def__: $x \in \ell^2(C_N)$ $\quad \mathcal{F}_d\, x = \gamma = \{\gamma_k\}_{k=0}^{N-1}$

where $\gamma_k = \sum_{n=0}^{N-1} x_n \exp\left(\frac{-2\pi i k n}{N} \right)$

$\mathcal{F}_d^{-1}$: $\quad x_k = \frac{1}{N} \sum_{m=0}^{N-1} \gamma_k \exp\left(\frac{2\pi i k m}{N} \right)$

This works because $\gamma_k = (x, e_k)$ $\qquad x = \sum \underbrace{\frac{(x, e_k)}{\|e_k\|^2}}_{\gamma_k} e_k$