Lecture 27

e_k, $e_k(n) = \exp\left\{\frac{2\pi i}{N} kn\right\}$

$$(e_k, e_j) = \underbrace{\sum_{n=0}^{N-1} \exp\left\{\frac{2\pi i}{N}(k-j)n\right\}}_{S} = \frac{1-\exp\{2\pi i(k-j)\}}{1-r} = 0$$

multiply each term by $\underbrace{\exp\left\{\frac{2\pi i}{N}(k-j)\right\}}_{\alpha}$ $\Rightarrow$ $\alpha S = S$ (b/c cyclicity) $\Rightarrow$ $\underline{S=0}$

$\{a_k\} \longleftrightarrow \sum a_k e^{ikt}$

$\sum a_k z^k$, $z = e^{it}$

$\{a_k\}$ has finitely many $\neq 0$ terms, then $\sum a_k z^k$ – trig polynomial

so take $\{a_k\}$ s.t. $a_k = 0$ for $k \geq N$, $k < 0$ $\Rightarrow$ $\sum a_k z^k$ – poly of deg N,

so it can be reconstructed by N points

Fast Fourier transform

C_N

$$X_k = \sum_{j=0}^{N-1} x_j e^{-\frac{2\pi i}{N}kj} \Rightarrow CN^2 \text{ operations}$$

$N = 2^k$

let $M = N/2$, $$X_k = \sum_{m=0}^{M-1} x_{2m} \exp\left\{-\frac{2\pi i}{N} 2mk\right\} + \sum_{m=0}^{M-1} x_{2m+1} \exp\left\{-\frac{2\pi i}{N}(2m+1)k\right\}$$

$$= \sum_{m=0}^{M-1} x_{2m} \exp\left\{-\frac{2\pi i}{M} mk\right\} + \left(\sum_{m=0}^{M-1} x_{2m+1} \exp\left\{-\frac{2\pi i}{M} mk\right\}\right) \exp\left\{-\frac{2\pi i}{N}k\right\}$$

$$= E_k + Odd_k \cdot \exp\left\{-\frac{2\pi i}{N} \cdot k\right\};$$

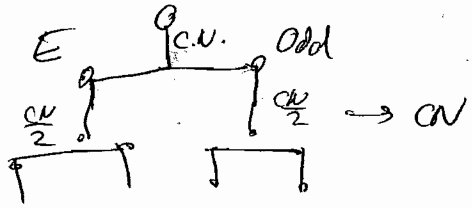
$0 \leq k \leq N-1$

$E_{k+M} = E_k$, $Odd_{k+M} = Odd_k$

$$= \begin{cases} E_k + Odd_k \cdot \exp\left\{-\frac{2\pi i k}{N}\right\} & 0 \leq k \leq M-1 \\ E_{k-M} - Odd_{k-M} \cdot \exp\left\{-\frac{2\pi i k}{N}\right\} & M \leq k \leq 2M-1 \end{cases}$$

– we change the sign b/c $\exp\left\{-\frac{2\pi i}{N}(k+M)\right\} = -\exp\left\{-\frac{2\pi i}{N}k\right\}$

Cost of Odd, Even + C·N



Cost of even = $C\frac{N}{2}$ + 2 costs of "halves"

$\Rightarrow CN \log_2 N$

Multiplication of polynomials

$= \sum_{k=0}^{n} a_k z^k \qquad g = \sum_{k=0}^{n} b_k z^k$

$$pg = \sum_{0}^{2n} c_k z^k, \qquad c_k = \sum_{j=0}^{k} a_j b_{k-j} = \sum_j a_j b_{k-j}$$

nonzero coefficients only in $0..2n$, so we could put them in a cyclic gp.

FT convolution $\to$ mult

$$\{a_k\}_0^{N-1} * \{b_k\}_0^{N-1} \longleftrightarrow \{A_k B_k\}_0^{N-1}$$

N^2 N operations

So mult can be done in $CN \log N$ steps

Cooley-Tookey algorithm, but created by Gauss

Fourier analysis on groups

Locally compact* abelian groups

*0 has a compact neighbourhood

Top gp $\to$ gp. operation cont.

- three types of gps: $\mathbb{R}, \mathbb{Z}, \mathbb{T}$

could be decomposed in $\mathbb{R}^k \oplus \mathbb{Z}^m \oplus \mathbb{T}^n$

Character

$\chi_{cont} : \Gamma \to \mathbb{T}$