Locally Compact Abelian Groups

$\mathbb{R}$ $\quad \mathbb{Z} \quad \mathbb{T}$

$\mathbb{R}^n \times \mathbb{Z}^m \times \mathbb{T}^k$

Character:

$\chi : \Gamma \to \mathbb{T}$

continuous

$\chi(a+b) = \chi(a) \cdot \chi(b)$.

All characters form a group.

$e: \quad \chi(x) = 1$.

Def] Group of characters is called dual group.

So characters of:

1.) $\mathbb{R}$: $\chi_a(x) = e^{iax}$. (continuous homomorphism).

2.) $\mathbb{Z}$: $\chi_\zeta(n) = (\zeta)^n = e^{i\theta n}$ $\quad \zeta = e^{i\theta}$ $\quad [\ \chi(1), \ \chi(n) = \chi(1)^n$

1.) $\hat{\mathbb{R}} \simeq \mathbb{R}$ since it can be parametrized by $\mathbb{R}$.

2.) $\hat{\mathbb{Z}} \simeq \mathbb{T}$

3.) $\mathbb{T}$ $\chi_n(z) = z^n$, $\hat{\mathbb{T}} = \hat{\mathbb{Z}}$.

$\hat{\hat{\Gamma}} = \Gamma$ $\qquad \hat{C}_n \cong C_n$ (cyclic group).

Haar Measure (translation invariant measure.)

Γ : $x \mapsto a + x$

$\mu(E) = \mu(a+E) \quad \forall a$

$\mathcal{F} : \Gamma \to \hat{\Gamma}$

$\mathcal{F}f(\xi) = \int f(x)\, \overline{\xi(x)}\, dx.$

Fourier Transform on $\mathbb{Z}$.

$x = \{x_k\}_{k \in \mathbb{Z}}$ with Haar Measure $:= \Sigma$.

$$\mathcal{F}x = \sum x_k \bar{z}^k. \qquad (\text{if } z = e^{it})$$

$$= \sum x_k e^{-ikt}.$$

Tempered Distributions

$$f(x) = \sum_0^{\hat{}} a_k x^k$$

$\hat{f}$

Schwartz Class $\mathcal{S}$.

$$\mathcal{S} : \{ f \in C^\infty(\mathbb{R}) \text{ s.t. } \|f\|_{m,n} = \|x^m f^{(n)}\|_{\ell^2} < \infty \quad \forall\, m,n \geq 0. \}$$

Remark

1. $\|(1+|x|)^m f^n\|_{\ell^2}$ is also finite, so we can look at different norms.

2.) can replace
$\|(1+|x|)^m f^{(n)}\|_{\ell^2}$ by $\|\_\_\|_p \quad 1 \leq p \leq \infty$

$$|\phi(x)| \leq \int_x^\infty |\phi'(x)|\, dx \quad \text{where } \phi \in C^1 \cap L^2.$$

To be true, $\phi(\infty) = 0$. $\quad \liminf_{x \to \infty} \phi(x) = 0.$

$$\leq \left(\int_{\mathbb{R}} |\phi'(x)|^2 (1+|x|)^2 \right)^{1/2} \cdot$$

finite constant. $\to \left(\int \frac{1}{(1+|x|)^2} \right)^{1/2}$

$\hookrightarrow \|(1+|x|)^m f^{(n)}\|_\infty$

$$\le C \|(1+|x|)^{m+1} f^{(n+1)}\|_2$$ i.e. decreases faster than any power.

- Convergence in $\mathcal{S}$:

$f_n \to f$ if $\|f_n - f\|_{k,m} \to 0 \quad \forall\, k, m.$

$$\|\mathcal{F} f\|_{n,m} = \|f\|_{mn} \qquad \mathcal{F}\mathcal{S} = \mathcal{S}$$

Define $\mathcal{S}'$ as continuous linear functionals on $\mathcal{S}$.

$$\varphi : \mathcal{S} \mapsto \mathbb{C}.$$

~~Formally~~ $\exists\, C$, s.t. $\forall f \in \mathcal{S}$ $\|\varphi(f)\| \le C \sum_{\text{finite}} \|f\|_{m,n}$

$\mathcal{S}'$ - tempered distributions.

Examples

$\varphi(f) = f(0)$. $\to$ Dirac delta function δ_0.

2.) $\varphi(f) = \int p(x) f(x)\, dx \qquad p(x)$ - polynomial.

$\psi \in C_0^\infty$

$$\varphi(f) = \int \hat{\psi}(\xi) f(\xi)\, d\xi = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(x) e^{-i\xi x}\, dx\, d\xi.$$

$$= \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi}} \int f(\xi) e^{-i\xi x}\, d\xi \right) \psi(x)\, dx.$$

$$= \int \psi(x) \hat{f}(x)\, dx.$$

Def $\varphi \in \mathcal{S}'$ then $\hat{\varphi}(f) = \varphi(\hat{f}) \quad \forall f \in \mathcal{S}$.

Exercises

Calculate $\hat{f}$

$\hat{f}(x) = x, x^2 \dots x^n$.