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Differentiating Distribution

Functions are also distributions.

Let $\psi \in C_0^\infty$

$$\varphi(h) = \int_{-\infty}^{\infty} \psi h \, dx \qquad h \in \mathcal{S}$$

Let $\psi \in C_0^\infty$

$$\dot{\varphi}(h) = \int_{-\infty}^{\infty} \psi'(x) h(x) \, dx \qquad \forall h \in \mathcal{S}$$

choose here $\psi'(x)dx = dv$

$$= \underbrace{\psi(x) h(x) \Big|_{-\infty}^{\infty}}_{0 \text{ since } \psi \in C_0^\infty} - \int_{-\infty}^{\infty} \psi(x) h'(x) dx = \varphi(-h')$$

<u>Def</u>: $\varphi \in \mathcal{S}'$ $\quad \varphi'$ is defined by $\varphi'(h) = -\varphi(h') \ \forall h \in \mathcal{S}$

$$\delta(h) = h(0) \qquad \delta'(h) = -\delta(h') = -h'(0)$$

<u>Recall</u>: the definition of derivative $\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

Then $\varphi_{\Delta x} = \dfrac{\delta(\cdot + \Delta x) - \delta(\cdot)}{\Delta x}$

$$\varphi_{\Delta x}(h) = \frac{h(-\Delta x) - h(0)}{\Delta x}$$

$$\Rightarrow \lim_{\Delta x \to 0} \varphi_{\Delta x}(h) = -h'(0)$$

Eigenvalues of Fourier Transform

Thm: $n \geq 0$ $\quad h_n := e^{x^2/2}\,[(e^{-x^2})^{(n)}]$

$$\hat{h}_n = (-i)^n h_n$$

Pf: Left as an exercise

Dual Space & Hahn-Banach Theorem

Def: $\mathfrak{X}$ is Banach. $\mathfrak{X}^*$ = All bounded linear functionals on $\mathfrak{X}$: $f : \mathfrak{X} \to \mathbb{F}$

Notation: $f \in \mathfrak{X}^*$ $\quad x \in \mathfrak{X}$ $\quad f(x) = \langle x, f \rangle$

Adjoint Operator: $\underset{\substack{\uparrow \\ \text{bounded}}}{A} : \mathfrak{X} \to Y$ then $A^* : Y^* \to \mathfrak{X}^*$ s.t. $\langle Ax, f \rangle = \langle x, A^* f \rangle$ $\forall x \in \mathfrak{X}$ $\forall f \in Y^*$.

More formally, for any fixed $f \in Y^*$ $\quad x \longmapsto \langle Ax, f \rangle$ is a bounded linear functional on $\mathfrak{X}$.

$\exists\, g \in \mathfrak{X}^*$ s.t. $\langle Ax, f \rangle = \langle x, g \rangle$ $\forall x \in \mathfrak{X}$. Call $g = A^* f$.

Hahn-Banach Thm: Let $\mathfrak{X}$ be normed, $L \subset \mathfrak{X}$ is a linear submanifold $f_0 : L \to \mathbb{F}$ s.t. $\|f_0(x)\| \leq C\|x\|$ $\forall x \in L$. Then $\exists$ $f : \mathfrak{X} \to \mathbb{F}$ s.t. $\|f(x)\| \leq C\|x\|$ $\forall x \in \mathfrak{X}$ and $f(x) = f_0(x)$ $\forall x \in L$.

Idea of Pf: Increase one dimension at a time works for finite or separable spaces. Use Zorn's Lemma for general case.

Observation: $|\langle x, f\rangle| \leq \|x\| \cdot \|f\|$

Cor: Given $x \in \mathcal{B}$ $\exists f \in \mathcal{B}^*$ s.t. $\|f\|=1$ and $f(x) = \langle x, f\rangle = \|x\|$

Alternatively: $\|x\| = \max\limits_{\substack{f \in \mathcal{B}^* \\ \|f\| \leq 1}} |\langle x, f\rangle| = \max\limits_{\substack{f \in \mathcal{B}^* \\ \|f\| \leq 1}} \operatorname{Re}\langle x, f\rangle$

Recall: $f \in \mathcal{B}^*$ is just a special linear operator, so $\|f\| = \sup\limits_{\|x\| \leq 1} |f(x)|$

Pf of Cor: $L = \{\lambda x : \lambda \in \mathbb{F}\}$

Define f_0 on L by $f_0(\lambda x) = \lambda \|x\|$

$|f_0(\lambda x)| = |\lambda| \|x\| = \|\lambda x\|$ so $\|f_0\| = 1$

By H-B extend f_0 to f. $f(\lambda x) = \lambda \|x\|$

$\|f\| \leq \|f_0\| = 1$

$\geq$ is trivial.

Ex $(\ell^p)^* = \ell^{p'}$ $\qquad (L^p)^* = L^{p'}$

$1 \leq p < \infty$

$\frac{1}{p} + \frac{1}{p'} = 1$ $\qquad$ In particular $(\ell^2)^* = \ell^2$

Dual of complex Hilbert space H is H with different linear structure.

because $\varphi(x) = (x, f) \Rightarrow (x, \alpha f) = \bar{\alpha}(x, f) = \bar{\alpha}\varphi(x)$