Lecture 30

$(\ell^p)^* = \ell^{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1$

<u>Pf.</u> $y = \{y_K\}_1^\infty \in \ell^{p'}$

$f(x) = \sum x_K y_K \qquad x = \{x_K\}_1^\infty \in \ell^p$

by <u>Hölder</u> $|f(x)| \le \|x\|_p \|y\|_{p'}$ and by <u>resonance lemma</u> $\|y\|_{p'} = \sup_{\|x\|_p \le 1} \left\{ \left|\sum x_K y_K\right| : \cancel{X} = \{x_K\}_1^\infty \in \ell^p \right\}$

$= \|f\|_{(\ell^p)^*}$

<u>Let</u> $f \in (\ell^p)^*$

$e_K = (0, \dots, 0, \underset{K}{1}, 0 \dots 0)$ std basis in ℓ^p

<u>Def.</u> $y_K = f(e_K)$

Then $f(x) = \sum x_K y_K$ for all finite sequences $x \in \ell^p$ (finite non-zero elem.)

By resonance lemma, $\|y\|_{\ell^{p'}} = \sup \left\{ \left|\sum x_K y_K\right| \cdot \cancel{y} = \{\cancel{x_K}\}_1^\infty \in \ell^p \right\} =$ $\|x\|_{\ell^p} \le 1$

$= \sup \left\{ \left|\sum x_K y_K\right| \cdot x = \{x_K\}_1^\infty \in \ell^p \text{ finite} \right\} = \sup \left\{ |f(x)| : x \in \ell^p, x\text{-finite} \right\} = \sup \left\{ |f(x)| : x \in \ell^p, \|x\|_{\ell^p} \le \right\}$ (finite seqs. are dense in ℓ^p)

$= \|f\|_{(\ell^p)^*}$

This works for $1 \le p < \infty$. Does not work for $p = \infty$ (finite seq. are not dense in ℓ^∞)

let $c_0 = \{x \in \ell^\infty : \lim x_K = 0\}$

c_0 - closed subspace of ℓ^∞ & finite sequences are dense in c_0

$\overset{\text{(similar pf)}}{\Rightarrow} \quad c_0^* = \ell^1$

$(\ell^\infty)^*$ – nobody knows

<u>Banach limits</u>

- consider $c_1 = \left\{ x = \{x_K\}_1^\infty \in \ell^\infty \text{ s.t. } \exists \text{ finite } \lim_{n \to \infty} x_n \right\}$

  c_1 - subspace (closed) of ℓ^∞

- define f_1 on c_1 as $f_1(x) = \lim_{n \to \infty} x_n$

  $|f(x)| \le \|x\|_\infty, \quad \|f_1\| \le 1$

On the other hand for $x = (1, 1, \dots 1, \dots)$, $f(x) = 1$, $\|x\|_\infty = 1$

$$|f_1(x)| = \|x\|_\infty \quad \text{so} \quad \|f_1\| = 1.$$

Extend f_1 to $f \in (\ell^\infty)^*$ by <u>Hahn - Banach</u>

$e_k = (0, 0, \dots \underset{k}{1}, 0 \dots 0 \dots)$

$f(x_k) = f_1(x_k) = 0$; <u>f - Banach limit</u>

$(L^p)^* = L^{p'}$ $\quad 1 \le p < \infty$

$\frac{1}{p} + \frac{1}{p'} = 1$

$C[a,b]^* = M[a,b]$

- complex finite Borel measures

Abstract measure $\quad \mathfrak{A}$ - σ algebra

$A \in \mathfrak{A} \Rightarrow A^c \in \mathfrak{A}$

$A_k \in \mathfrak{A} \Rightarrow \bigcup A_k \in \mathfrak{A}$

$A_k \in \mathfrak{A} \Rightarrow \bigcap A_k \in \mathfrak{A}$

$\mu : \mathfrak{A} \to [0, \infty]$ s.t. $\mu\left(\bigcup_k A_k\right) = \sum \mu(A_k)$

if $A_k \cap A_j = \emptyset \quad \forall j \neq k$

$(L^\infty_{[a,b]})^* = ?$ $\quad \subset C[a,b]^* = M$

$$\boxed{\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4}$$

$$\mu(A) = \int_A f(x)\,dx \qquad f \in L'_\infty$$

<u>Dual extremal problems</u>

dist from x to L $\quad \operatorname{dist}(x, L) = \inf \{ \|x - y\| : y \in L \}$

<u>L.</u> Let $x \in X$, $L_{\text{subspace}} \subset X$, $d = \operatorname{dist}(x, L)$

Then $d = \max \{ |f(x)| \;\; f \in X^* \text{ s.t } f_{|L} = 0 \text{ and } \|f\| \le 1 \}$

$\operatorname{Re} f(x)$ $\qquad \|f\| = 1$