Lecture 32

Prop. if $\{x_n\}_1^\infty$ - complete $\Leftrightarrow$ $\mathcal{L}(x_K : K \geq 1)$ dense in X $\Rightarrow$ $\{x_n'\}_1^\infty$ is unique

Pf. x_n' ; let $\varphi \in X^*$ s.t. $\varphi(x_K) = 0 \ \forall K \neq n$

Then φ is completely det. by $\varphi(x_n)$

$\varphi(\sum c_j x_j) = c_n \varphi(x_n) \Rightarrow \varphi \ \ \varphi(x_K) = 0 \ \forall K \neq n$ is 1-dim.

$\Rightarrow \exists! \ \varphi$ s.t. $\varphi(x_K) = \delta_{n,K} \Rightarrow x_n'$ - unique

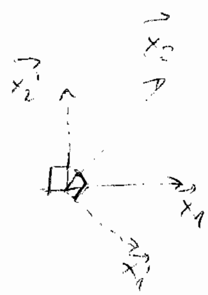
If $\operatorname{span}\{x_K : K \geq 1\} \neq X$ we can add to any x_n' a vector $\varphi_n \in X^*$ s.t. $\varphi_n(x_K) = 0 \ \forall K$

$\{x_n\}, \{x_n'\}$ minimal biorthogonal

If $\vec{x} = \sum_{\text{finite}} c_K x_K \Rightarrow c_K = \langle x, x_K' \rangle$

$x \sim \sum \langle x, x_K' \rangle x_K$

Formal Fourier series



$\vec{x} = \langle \vec{x}, \vec{x}_1' \rangle \vec{x}_1 + \langle \vec{x}, \vec{x}_2' \rangle \vec{x}_2$

Banach basis theorem

let $\{x_n\}_1^\infty$ L.I. & complete in X - Banach

One can define $P_n \ \mathcal{L}(x_K : K \in \mathbb{N}) \to \mathcal{L}(x_K : K \in \mathbb{N})$

$P_n(\sum c_K x_K) = \sum_{K=1}^{n} c_K x_K$ (proj on the first n coord.)

$\downarrow$ finite if only L.I.

Rmk if $\{x_n\}$ - minimal then $P_n x = \sum_{K=1}^{n} \langle x, x_K' \rangle x_K$

Thm Let $\{x_n\}_1^\infty$ be a complete L.I. syst. in X. Then $\{x_n\}$ - basis $\Leftrightarrow \exists \ C < \infty$ s.t. $\|P_n\| \leq C$

Pf. Assume that $\{x_n\}$ - minimal. Then P_n are bounded operators on X

$\{x_n\}_1^\infty$ is a basis $\Leftrightarrow \sum \langle x, x_n' \rangle x_n$ converges $\forall x \in X \Leftrightarrow P_n x \to x \ \forall x \in X \Longrightarrow$

$\underset{\text{uniform bddness.}}{\Longrightarrow} \|P_n\| \leq C < \infty$

$\Longleftarrow$ convergence $P_n x \to x$ $\forall x = \sum_{\text{finite}} c_K x_K$ & apply $\frac{\varepsilon}{3}$ thm.

$\{x_n\}_1^\infty$ in $\mathcal{H}$ – Hilbert

$\{x_n\}_1^\infty$ – complete

If $\{x_n\}$ – O.N.B. then $\|\vec{x}\|^2 = \sum |(x, x_n)|^2$ $\forall x$

$$\left\|\sum c_k \vec{x}_k\right\|^2 = \sum |c_k|^2$$

Def. a complete system $\{x_n\}_1^\infty$ is called a Riesz basis if $\exists\, C < \infty$ s.t.

$$\frac{1}{C}\sum|c_k|^2 \overset{①}{\le} \left\|\sum c_k x_k\right\|^2 \overset{②}{\le} C \underbrace{\sum |c_k|^2}_{\|Tx\|^2}$$

$$\left\|\sum c_k x_k\right\|^2 \asymp \sum |c_k|^2 \qquad \approx$$

Prop. $\{x_n\}_1^\infty$ – Riesz basis in $\mathcal{H}$ iff $\exists$ invertible $T: \mathcal{H} \to \mathcal{H}_1$ s.t. $\{Tx_n\}_1^\infty$ – O.N.B.

$\Updownarrow$

$\exists\, R: \mathcal{H}_1 \to \mathcal{H}$ (T – orthogonalizer of $\{x_n\}$)

O.N.B. $\{e_n\} \in \mathcal{H}_1$ s.t. $x_n = Re_n$

Riesz basis is a basis and moreover an uncond basis $x = \sum c_k x_k$ converges uncond. (any order)

Pf of prop. ① $\iff$ T – bdd.

② $\iff$ T^{-1} – bdd.

– let $x_n = Re_n$

$$\boxed{x_n' = (R^*)^{-1} e_n}$$

Pf $(x_n, x_k') = \left(Re_n, (R^{-1})^* e_k\right) = \left(R^{-1}Re_n, e_k\right) = (e_n, e_k) = \delta_{n,k}$

Cor. $\{x_n\}$ is Riesz basis $\iff$ $\{x_n'\}$ – R.B.

Prop. $\{x_n\}$ – R.B. $\iff$ $\exists\, C \in (0,\infty)$ s.t. $\frac{1}{C}\|x\|^2 \le \sum |(x, x_n)|^2 \le C\|x\|^2$ $(*)$

Pf $(*)$ means that $\{x_n'\}$ is R.B.

– let $\{e^{int}\}_{n\in\mathbb{Z}}$ in $L^2_{2\pi}(w)$ is R.B $w, \frac{1}{w} \in L^\infty$

$T: L^2_{2\pi}(w) \to L^2$ $\quad Tf = f$ is orthog.

$L^2_{2\pi}$

$e^{i\lambda_k t} = x_n(t)$, $\lambda_k \in \mathbb{R}$

$$\sup_{k \in \mathbb{Z}} |\lambda_k - k| \le \frac{1}{4}$$

Then $\{x_n\}$ is Riesz (Kadet $\frac{1}{4}$ thm)