Example

$L^2_{2\pi}(w)$

$$\|f\|^2 = \int_0^{2\pi} |f(x)|\, \omega(x)\, \frac{dx}{2\pi}$$

$e_n(x) = e^{inx}$ $\quad n \in \mathbb{Z}$.

1.) $\{e_n\}$ is linearly independent iff $\omega \not\equiv 0$.

$$\sum_{k=-N}^{N} c_k e_k = 0 \iff \sum \|\sum_{-N}^{N} c_k e_k\|^2 = 0$$

$$\iff \int_0^{2\pi} |\sum_{-N}^{N} c_k e^{ikx}|\, \omega(x)\, \frac{dx}{2\pi} = 0$$

$$\iff \underbrace{|\sum c_k e^{ikx}|^2}\, \omega(x) = 0.$$

can be zero at finitely many points. $\leq 2N$

$$|\sum_{-N}^{N} c_k e^{ikx}| = 0 \iff |\sum_{k=0}^{2N} c_{k-N} e^{ikx}| = 0.$$

$\sum_{k=0}^{2N} C_{k-N} z^k$ has $2N$ zeros in $\mathbb{C}$.

2.) $\{e_n\}_{n\in\mathbb{Z}}$ is minimal iff $\frac{1}{\omega} \in L^1$. (Kolmogorov).

$\operatorname{dist}(e_0, \mathcal{L}(e_k : k \neq 0))$ $\quad$ Claim
d in $L^2_{2\pi}(\omega)$

$$= 1/\|e_0^*\| = 1/\int 1 \frac{1}{\omega(x)} \frac{dx}{2\pi}.$$

Thm: $\operatorname{dist}(x, E) = \max\{|\phi(x)| : \phi \in E^*, \phi|_E = 0\}$.
(Real)

$E = \text{span}\{e_k : k \neq 0\}$

$x \in e_0$ $\qquad \varphi|_E = 0 \iff \Phi(e_k) = 0 \ \forall\, k = 0$

$\Rightarrow \quad \phi = \alpha e_0'$.

$\|\phi\| \leq 1 \quad \Rightarrow \quad |\alpha| \leq \dfrac{1}{\|e_0'\|_1}$

So. $|\varphi(e_0)| = |\alpha \langle e_0, e_0' \rangle| = |\alpha|$.

Note: $e_k'(x) = e^{-ikx}$ in $L^2_{2\pi}(\frac{1}{\omega})$

when $\langle f, g \rangle = \int f(t)\, g(t)\, \frac{dt}{2\pi}$.

3.) $\{e_n\}$ is Riesz Basis in $L^2_{2\pi}(\omega)$

iff ω and $\frac{1}{\omega}$ are bounded. i.e. $\in L^\infty$

4.) $\{e_n\}$ is a basis iff:

$$\sup_I \frac{1}{|I|} \int \omega(x) \frac{dx}{2\pi} \; \frac{1}{|I|} \int \frac{1}{\omega(x)} \frac{dx}{2\pi} \quad \text{is finite.}$$

Example $\quad$ Monomials x^n in $L^2(a,b)$ are "bad".

Not minimal.

$L^2(0,1)$

Say x can be approximated by $\sum_{k \neq 1} c_k x_k$.

Note: $\sqrt{t}$ can be approx. in $C[0,1]$ and so in $L^2(0,1)$.

by polynomials $\sum c_k t^k$.

Change variables : $t = x^2 \qquad x \sim \sum c_k x^{2k}$.

$\Rightarrow$ So $\text{dist}(x, \mathcal{L}\{x^k : k \neq 1\}) \leq \text{dist}(x, \mathcal{L}\{x^{2k} : k \geq 0\}) = 0$

Frames

Def $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$. is called a frame

if $\exists$ C s.t. $\forall x \in \mathcal{H}$ $\sum |\langle x, x_n\rangle|^2 \leq C\|x\|^2$
$\geq \frac{1}{C} \|x\|^2$

A frame ~~is minimal~~ and minimal system is the same as a Riesz basis.

Reminder: x_n is Riesz basis iff $\exists$ $\{e_n\}$ ONB and invertable R s.t. $x_n = Re_n$.

$$x_n' = (R^*)^{-1} e_n.$$

$$\{x_n\} \text{ is RB} \iff \{x_n\}^{\prime} \text{ is RB}.$$

$$\{x_n\}' \text{ is RB} \iff \|\underbrace{\sum c_k x_k'}_{x.}\|^2 \leq C \sum |c_k|^2$$
$$\geq \frac{1}{c} |c_k|^2$$

$$c_k = \langle x, x_k\rangle$$

$$\frac{1}{c}\|x\|^2 \leq \sum |\langle x, x_k\rangle|^2 \leq C\|x\|^2$$

Ex $\{f_n\}$ RB.
$\{g_n\}$ RB.

$x_{2n} = f_n$ $\quad x_{2n+1} = g_n$.

L^2_π $\quad e_n(x) = e^{inx}$. $\quad n \in \mathbb{Z}$.

frame but not a basis.

or L^2_r where $r < 2\pi$.