11/21

<u>Feightinger Conjecture</u>: Let $\{f_n\}_{n=1}^{\infty}$ be a frame s.t. $\frac{1}{B} \leq \|f_n\| \leq B$ (Can assume that $B=1$)

The conjecture is that $\{f_n\}$ is a finite union of Riesz sequences.

<u>Def</u>: A <u>Riesz sequence</u> is a Riesz basis in its closed linear span.

This conjecture follows from the Paving Conjecture.

<u>Paving Conjecture</u>: Let $A: \ell^2 \to \ell^2$ with 0 diagonal, $\|A\|=1$, Given $0<\varepsilon<1$

$\exists$ $N=N(\varepsilon)$ s.t. $\forall$ A as above $\exists$ $\sigma_1, \dots, \sigma_n$ disjoint decomposition of the set of indices s.t. $A_n = \{a_{jk}\}_{j,k \in \sigma_n}$ $\|A_n\| \leq \varepsilon$. $\forall n = 1, 2, \dots, N$.

<u>Pf of Feightingers Given Paving</u>: If $\frac{1}{B} \leq \|f_n\| \leq B$ $\forall n$ $\{f_n\}$ is a frame, then $\{\frac{f_n}{\|f_n\|}\}$ is also a frame, so WLOG $\|f_n\|=1$

If $\{f_n\}$ is a frame $\frac{1}{C}\|x\|^2 \leq \sum |(x,f_n)|^2 \leq C\|x\|^2$

$\Rightarrow \forall \{c_n\} \in \ell^2$ $\quad \|\sum c_n f_n\| = \sup_{\|x\| \leq 1} |\sum (c_n f_n, x)| = \sup_{\|x\|\leq 1} |\sum c_n (f_n, x)|$

$$\leq \sup_{\|x\|\leq 1} \|\{c_n\}_1^{\infty}\|_{\ell^2} \left(\sum |(f_n,x)|^2\right)^{1/2} \quad \text{(Cauchy-Schwarz)}$$

$$\leq \sup_{\|x\|\leq 1} \|\{c_n\}_1^{\infty}\|_{\ell^2} \, C\|x\| \quad \text{(def. of frame)}$$

$$\Rightarrow \|\sum c_n f_n\| \leq C \|\{c_n\}\|_{\ell^2}$$

As a Riesz sequence, $\frac{1}{C}\sum |c_k|^2 \leq \|\sum c_k f_k\|^2 \leq C \sum |c_k|^2$

$\|\sum c_n f_n\|^2 = \sum_{j,k} B_{jk} c_k \bar{c}_j$ $\quad$ where $B_{jk} = (f_k, f_j)$

$$(B\{c_n\}, \{c_n\}) \leq C \|\{c_k\}_1^{\infty}\|_{\ell^2}^2$$

We want to split $\{f_n\}$ into N Riesz sequences.

Split $\mathbb{N} = \sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_N$ pairwise disjoint

$$\frac{1}{C} \sum_{k \in \sigma_n} |c_k|^2 \le \sum_{j,k \in \sigma_n} B_{jk} c_k \bar{c}_j \le C \sum_{k \in \sigma_n} |c_k|^2$$

True if $\{B_{jk}\}_{j,k \in \sigma_n}$ is invertible (call this B^σ)

$A = B - I$ has zeros on the diagonal

B^σ is invertible if $\|A^\sigma\| < 1$

So Feichtinger follows from Paving for $\varepsilon < 1$.

<u>Invertibility</u>: $B^\sigma = B^{\sigma *}$, $(B^\sigma x, x) \geq 0$

$$(B^\sigma x, x) \le C \|x\|^2$$

$$(B^\sigma x, x) = (\underbrace{(B^\sigma)^{1/2} x}_{y}, \underbrace{(B^\sigma)^{1/2} x}_{y})$$

$$(x,x) = ((B^\sigma)^{-1/2} y, (B^\sigma)^{-1/2} y) \le C (y,y) = C((B^\sigma)^{1/2} x, (B^\sigma)^{1/2} x) = C(B^\sigma x, x)$$

<u>Ex</u>: If $A = A^*$ and $(Ax, x) \geq 0$ then $\sigma(A) \subset [0, \infty)$