# Lecture 35

## Wavelets

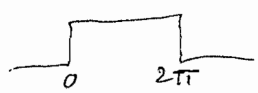
### Basis in $L^2(\mathbb{R})$

Disadvantages of Fourier transform
- does not handle well finite signals (inf. frequencies)
- no good description of $f \in L^1$
  $f \in C \cap L^2$
  $f \in$ a lot of other natural spaces in terms of $\hat{f}$
- want to have countable basis

Gabor bases

$e_k(x) = e^{ikx}$ ONB in $L^2\left([0, 2\pi)\ \frac{dx}{2\pi}\right)$

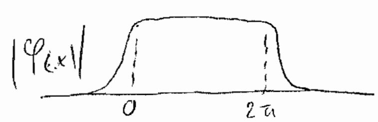
$e_k^0(x) = e^{ikx} \chi_{[0,2\pi)}(x)$




$e_k^n(x) = e_k^0(x - 2\pi n)$

$\{e_k^n\}_{k,n \in \mathbb{Z}}$ - ONB in $L^2\left(\mathbb{R}\ \frac{dx}{2\pi}\right)$

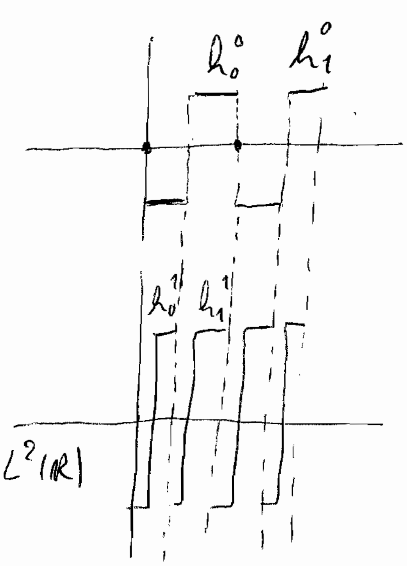
$e_k^0(x) = e^{ikx} \varphi(x)$




$\boxed{\text{Ex:}}$ Is it possible to get ONB with $\varphi \neq \chi_{[0,2\pi)}$ ?

Haar "wavelet" basis

$$h(x) = \begin{cases} -1 & 0 \le x < \frac{1}{2} \\ 1 & \frac{1}{2} \le x < 1 \\ 0 & x \notin [0,1) \end{cases}$$




$h_k^n(x) = h(2^n x - k) \cdot 2^{\frac{n}{2}}$

Thm $h_k^n$ $k, n \in \mathbb{Z}$ is ONB in $L^2(\mathbb{R})$

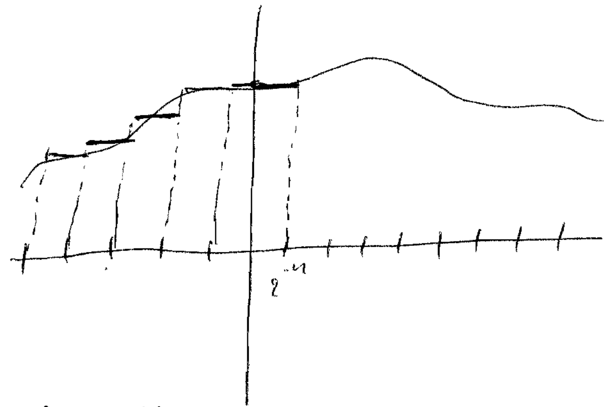
Pf ONS - trivial

why complete?

Introduce operators $E_n$

$$E_n f(x) = \frac{1}{|I|}\int_I f(x)\,dx \quad \text{where } I = [2^{-n}k, 2^{-n}(k+1)) \ni x$$

$[2^{-n}k, 2^{-n}(k+1))$ dyadic intervals




$$E_n f - E_m f = \sum_{\substack{j\in\mathbb{Z} \\ m\le k<n}} (f, h_k^j)\, h_k^j$$

$\boxed{n>m}$

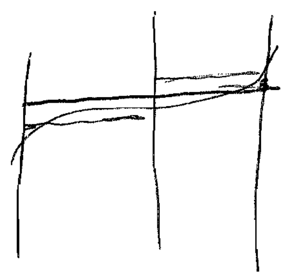
To prove, sufficient to consider $n=m+1$

$E_m f$ - constant on $[2^{-m}k, 2^{-m}(k+1))$

$n=m+1$

$E_n f$ - constant on $[2^{-n}k, 2^{-n}(k+1))$

$$\int_I (E_n f - E_m f)\,dx = 0 \quad \forall I = [2^{-m}k, 2^{-m}(k+1))$$



So $E_n f - E_m f = \sum c_k h_k^m$ $\quad \underline{n=m+1}$

$$c_k = ((E_n f - E_m f), h_k^m) = (E_n f, h_k^m) = (f, h_k^m)$$

(*) $E_n f \to f$ in $L^2$ as $n\to\infty$

$E_n f \to 0$ in $L^2$ as $n\to-\infty$ $\quad \Rightarrow \sum_{n,k\in\mathbb{Z}} (f, h_k^n)\, h_k^n = f$

To prove (*) we use $\frac{\varepsilon}{3}$ thm.

If $f\in C_0 \Rightarrow E_n f \xrightarrow[n\to\infty]{} f$ in $L^2$ and $E_n f \xrightarrow[n\to-\infty]{} 0$ in $L^2$

supp $f\subset[a,b]$

Then for $n\ge 0$ supp $E_n f\subset[a-1,b+1]$

$f$ - unif cont on $[a-1,b+1] \Rightarrow E_n f \rightrightarrows f$ as $n\to\infty$ on $[a-1,b+1] \Rightarrow E_n f\to f$ in $L^2$

Let $\operatorname{supp} f \subset [-2^N, 2^N]$

For $n > N$, $|E_{-n} f| \leq \chi_{[-2^n, 2^n]} \cdot 2^{-n} \int_{-2^N}^{2^N} |f| \, dx$

$$\Downarrow$$

$$\|E_{-n} f\|_{L^2}^2 \leq 2^{-2n} \left( \int \quad \right)^2 \cdot 2 \cdot 2^n \underset{n \to +\infty}{\longrightarrow} 0$$

$E_n$ are orthogonal projections, so $\|E_n\| = 1$ . uniformly bdd. , so $\frac{\varepsilon}{3}$ thm applies.