Multiresolution analysis (MRA)

E_n operators.

Def MRA is a collection of subspaces V_n (think of Ran E_n)

= functions in L^2 constant on dyatic intervals of size 2^{-n}. $n \in \mathbb{Z}$.

Properties

1.) $V_n \subset V_{n+1}$ $\forall n \in \mathbb{Z}$.

2.) $f \in V_j \iff f(2x) \in V_{j+1}$.

3.) $\bigcap V_j = \{0\}$.

4.) $Cl\left(\bigcup_{j \in \mathbb{Z}} V_j\right) = L^2(\mathbb{R})$.

5.) $\exists$ $\varphi \in V_0$ s.t. φ_k ($\varphi_k(x) = \varphi(x-k)$.) form an ONB in V_0. $(n \in \mathbb{Z})$

(5') φ_k $k \in \mathbb{Z}$ is a Riesz basis.

Remark For Haar system $\varphi = \chi[0,1)$.

Theorem $5, 2, 1 \Rightarrow 3$ or $5', 2, 1 \Rightarrow 3$.

Prop: Let $f \in L^2(\mathbb{R})$ and $f_k(x) = f(x-k)$.

Then f_k is orthonormal system iff

$$\sum_{n \in \mathbb{Z}} |\hat{f}(\xi + 2\pi n)|^2 \equiv 2\pi$$

Pf

$$\hat{f}_k(\xi) = \frac{1}{\sqrt{2\pi}} \int f(x-k) e^{-i\xi x} \, dx$$
$$= \frac{1}{\sqrt{2\pi}} \int f(x-k) e^{-i\xi(x-k)} \, dx \cdot e^{-i\xi k}.$$
$$= e^{-i\xi k} \hat{f}(\xi).$$

Fourier T. ~~[illegible]~~ is unitary so need to check when $\underbrace{e^{-ik\xi}\hat{f}(\xi)}_{g_k}$ $k \in \mathbb{Z}$ is ONB.

$$\langle g_k, g_j \rangle = \int e^{i(j-k)\xi} |\hat{f}(\xi)|^2 d\xi.$$
$$= \sum_{n\in\mathbb{Z}} \int_{2\pi n}^{2\pi(n+1)} e^{i(j-k)} |\hat{f}(\xi)|^2 d\xi = \int_0^{2\pi} e^{i(j-k)\xi} \underbrace{\sum |\hat{f}(\xi + 2\pi n)|^2}_{\Phi} d\xi.$$

Then g_k ONS $\iff$ $\hat{\Phi}(k) = \begin{cases} 0 & k \neq 0 \\ 2\pi & k = 0 \end{cases}$ $\iff$ $\Phi \equiv 2\pi.$

Constructing Wavelets from MRA

Define $W_0 = V_1 \ominus V_0 = \{ x \in V_1 : x \perp V_0 \}.$

Then $V_1 = V_0 + W_0$. So $L^2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}} W_n$ $\quad$ $V_k = V_{k+1} \ominus V_k$

$\Uparrow$ $\quad$ so $V_{k+1} = V_k \oplus W_k.$

If $n > m$

$$V_n \ominus V_m = \bigoplus_{m\le k<n} W_k$$

Remark

Property 2 $\Rightarrow$ $f \in W_k \iff f(2x) \in W_{k+1}$.

If we found ψ s.t. $\psi_k : \psi_k(x) = \psi(x-k)$ $k \in \mathbb{Z}$. forming ONB is W, then we constructed wavelets.

$\psi_k^n(x) = 2^{n/2} \psi(2^n x - k)$. is ONB in $L^2(\mathbb{R})$.

ψ - mother wavelength.

From $\phi \to \psi$.

$\phi \in V_0 \qquad \frac{1}{2}\phi\left(\frac{x}{2}\right) \in V_{-1} \subset V_0$

$\Rightarrow \quad \frac{1}{2}\phi\left(\frac{x}{2}\right) = \sum \alpha_k \phi(x+k).$

$\alpha_k = \frac{1}{2}\int \phi\left(\frac{x}{2}\right)\overline{\phi(x+k)}\,dx.$

$$\hat{\phi}(2\xi) = \hat{\phi}(\xi)\cdot\underbrace{\sum \alpha_k e^{ik\xi}}_{m_0(\xi)\ \to\ \text{low pass filter.}}$$

$\phi \longrightarrow m_0$

$|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 \equiv 1 \implies \boxed{\text{Exercise}}$

or some constant.