

Multiresolution analysis (MRA)

E_n operators.

Def MRA is a collection of subspaces V_n (think of $\text{Ran } E_n$)
 = functions in L^2 constant on dyadic intervals of size 2^{-n} $n \in \mathbb{Z}$.

Properties

1.) $V_n \subset V_{n+1} \quad \forall n \in \mathbb{Z}$.

2.) $f \in V_j \iff f(2x) \in V_{j+1}$.

3.) $\bigcap V_j = \{0\}$.

4.) $\text{cl}\left(\bigcup_{j \in \mathbb{Z}} V_j\right) = L^2(\mathbb{R})$.

5.) $\exists \varphi \in V_0$ s.t. φ_k ($\varphi_k(x) = \varphi(x-k)$)
 form an ONB in V_0 . ($n \in \mathbb{Z}$)

(5') φ_k $k \in \mathbb{Z}$ is a Riesz basis.

Remark For Haar system. $\varphi = \chi_{[0,1)}$.

Theorem 5, 2, 1 \implies 3 or 5', 2, 1 \implies 3.

Prop: Let $f \in L^2(\mathbb{R})$ and $f_k(x) = f(x-k)$.

Then f_k is orthonormal system ~~iff~~ iff

$$\sum_{n \in \mathbb{Z}} \left| \int_{\mathbb{R}} f(x) \overline{f(x-2\pi n)} dx \right|^2 = 2\pi$$

pf

$$\begin{aligned}\hat{f}_k(\xi) &= \frac{1}{\sqrt{2\pi}} \int f(x-k) e^{-i\xi x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int f(x-k) e^{-i\xi(x-k)} dx \cdot e^{-i\xi k} \\ &= e^{-i\xi k} \hat{f}(\xi).\end{aligned}$$

Fourier T. ~~operator~~ is unitary so need to check when $\underbrace{e^{-i\xi k} \hat{f}(\xi)}_{g_k} \quad k \in \mathbb{Z}$ is ONB.

$$\begin{aligned}\langle g_k, g_j \rangle &= \int e^{i(j-k)\xi} |\hat{f}(\xi)|^2 d\xi \\ &= \sum_{n \in \mathbb{Z}} \int_{2\pi n}^{2\pi(n+1)} e^{i(j-k)\xi} |\hat{f}(\xi)|^2 d\xi = \int_0^{2\pi} e^{i(j-k)\xi} \underbrace{\sum |\hat{f}(\xi + 2\pi n)|^2}_{\Phi} d\xi.\end{aligned}$$

$$\text{Then } g_k \text{ ONS} \iff \hat{\Phi}(k) = \begin{cases} 0 & k \neq 0 \\ 2\pi & k = 0 \end{cases} \iff \Phi \equiv 2\pi.$$

Constructing Wavelets from MRA

$$\text{Define } W_0 = V_1 \ominus V_0 = \{x \in V_1 : x \perp V_0\}.$$

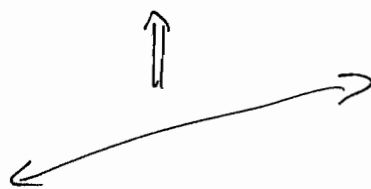
$$\text{Then } V_1 = V_0 \oplus W_0. \quad \text{So } L^2(\mathbb{R}) = \bigoplus_{n \in \mathbb{Z}} W_n$$

$$V_k = V_{k+1} \ominus V_k$$

$$\text{so } V_{k+1} = V_k \oplus W_k.$$

If $n > m$

$$V_n \ominus V_m = \bigoplus_{m+1 \leq k \leq n} W_k$$



Remark

$$\text{Property 2} \Rightarrow f \in W_k \iff f(2x) \in W_{k+1}.$$

If we found ψ s.t. $\psi_k : \psi_k(x) = \psi(x-k) \quad k \in \mathbb{Z}$. forming ONB in W , then we constructed wavelets.

$$\psi_k^n(x) = 2^{n/2} \psi(2^n x - k). \quad \text{is ONB in } L^2(\mathbb{R}).$$

Ψ - mother wave length.

From $\phi \rightarrow \psi$.

$$\phi \in V_0 \quad \frac{1}{2} \phi\left(\frac{x}{2}\right) \in V_{-1} \subset V_0$$

$$\Rightarrow \frac{1}{2} \phi\left(\frac{x}{2}\right) = \sum \alpha_k \phi(x+k).$$

$$\alpha_k = \frac{1}{2} \int \phi\left(\frac{x}{2}\right) \overline{\phi(x+k)} dx.$$

$$\hat{\phi}(2\xi) = \hat{\phi}(\xi) \cdot \underbrace{\sum \alpha_k e^{ik\xi}}_{m_0(\xi)} \rightarrow \text{low pass filter.}$$

$$\phi \rightarrow m_0$$

$$|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 \equiv 1 \quad \Rightarrow \quad \boxed{\text{Exercise}}$$

↳ or some constant.