Lecture 37

Prop. $f_k(x) = f(x-k)$ is ONS then $\sum \left|\hat{f}(\xi + 2\pi n)\right|^2 = \frac{1}{2\pi}$ — Was in last class

$|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 \equiv 1 \leftarrow$ Ex

Pf. $$\frac{1}{2\pi} = \sum_{n\in\mathbb{Z}} \left|\hat{\varphi}(2\xi + 2\pi n)\right|^2 = \sum_{n\in\mathbb{Z}} \left|\hat{\varphi}(\xi+\pi n)\cdot m_0(\xi+\pi n)\right|^2 =$$

$$= \Big(\underbrace{\sum_{k\in\mathbb{Z}} \left|\hat{\varphi}(\xi + 2\pi k)\right|^2}_{\frac{1}{2\pi}}\Big)|m_0(\xi)|^2 + \Big(\underbrace{\sum_{k\in\mathbb{Z}} \left|\hat{\varphi}(\xi + \pi + 2\pi k)\right|^2}_{\frac{1}{2\pi}}\Big)|m_0(\xi+K)|^2$$

$m_0(\xi) = m_0(\xi + 2\pi)$

Lemma a) $V_0 = \{ f : \hat{f}(\xi) = \hat{\varphi}(\xi)\cdot \ell(\xi),\ \ell \in L^2_{2\pi} \}$

b) $V_n = \{ f : \hat{f}(\xi) = \hat{\varphi}(2^{-n}\xi)\, \ell(2^{-n}\xi),\ \ell \in L^2_{2\pi} \}$

Pf a) $f \in V_0 \iff f = \sum c_k \varphi_k$, $\sum |c_k|^2 < \infty$

$$\iff \hat{f}(\xi) = \sum_{k\in\mathbb{Z}} c_k \hat{\varphi}(\xi) e^{-ik\xi} = \hat{\varphi}(\xi)\cdot \underbrace{\sum_{k\in\mathbb{Z}} c_k e^{-ik\xi}}_{\ell}$$

b) $\widehat{f(ax)}(\xi) = \hat{f}\left(\frac{\xi}{a}\right)$ (if we do scaling → f.t. ⇒ opposite scaling)

Cor. $V_{-1} = \{ f \in L^2 : \hat{f}(\xi) = m_0(\xi)\hat{\varphi}(\xi) m(2\xi),\ m \in L^2_{2\pi} \}$

follows from: $\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi)$
& b)

Cor. let φ from MRA. $\hat{\varphi}$ cont at 0 and $\hat{\varphi}(0) \neq 0$

Then $Cl\left(\bigcup_{n\in\mathbb{Z}} V_n\right) = L^2$ $\boxed{\text{Ex.}}$ (Hint: use b) & f.t.)

Want to find $\psi \in W_0 = V_1 \ominus V_0$ s.t. ψ_k $k \in \mathbb{Z}$ is ONB in W_0

Need to describe W_0. More convenient to describe W_{-1} and get W_0 by scaling.

$$W_{-1} = \{ f \in V_0 \text{ s.t. } f \perp V_{-1} \} = \{ f : \hat{f}(\xi) = \hat{\varphi}(\xi)\, l(\xi) \,,\ f \perp V_{-1} \}$$

F.t. - unitary $\Rightarrow$ can check orthogonality in F.t.

$$f \perp V_{-1} \iff \int_{\mathbb{R}} l(\xi)\hat{\varphi}(\xi)\overline{\hat{\varphi}(\xi)}\,\overline{m_0(\xi)}\,\overline{m(2\xi)}\,d\xi = 0 \quad \forall m \in L^2_{2\pi}$$

$$(*) \quad \int_0^{2\pi} l(\xi)\,\overline{m_0(\xi)\, m(2\xi)}\,d\xi = 0 \quad \forall m \in L^2_{2\pi}$$

$$\text{b/c} \quad \int_{\mathbb{R}} l(\xi)|\hat{\varphi}(\xi)|^2\overline{m_0(\xi)\,m(2\xi)}\,d\xi = \sum \int_0^{2\pi} l(\xi)|\hat{\varphi}(\xi+2\pi k)|^2\,\overline{m_0(\xi)m(2\xi)}\,d\xi$$

$$= \int_0^{2\pi} \underbrace{\sum |\hat{\varphi}(\xi+2\pi k)|^2}_{\frac{1}{2\pi}}\,\overline{m_0(\xi)\, m(2\xi)}\,d\xi$$

$m(2\xi)$ - π periodic

$$(*) \iff \int_0^{\pi} \overline{m}(2\xi)\left(l(\xi)\overline{m_0(\xi)} + l(\xi+\pi)\overline{m(\xi+\pi)} \right) d\xi = 0$$

$$\Updownarrow$$

$$l(\xi)\overline{m_0(\xi)} + l(\xi+\pi)\overline{m(\xi+\pi)} \equiv 0$$

$$\Updownarrow$$

$$\begin{pmatrix} l(\xi) \\ l(\xi+\pi) \end{pmatrix} = \lambda(\xi)\begin{pmatrix} -\overline{m_0(\xi+\pi)} \\ \overline{m_0(\xi)} \end{pmatrix}$$

never 0 b/c have ex $|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1$

- Substitute $\xi = \pi + \eta$

$$\begin{pmatrix} l(\eta+\pi) \\ l(\eta+2\pi) \end{pmatrix} = \lambda(\eta+\pi)\begin{pmatrix} -\overline{m_0(\eta)} \\ \overline{m_0(\eta+\pi)} \end{pmatrix} \rightarrow \begin{pmatrix} l(\xi) \\ l(\xi+\pi) \end{pmatrix} = \lambda(\xi+\pi)\begin{pmatrix} \overline{m_0(\xi+\pi)} \\ -\overline{m_0(\xi)} \end{pmatrix} \Rightarrow$$

change order of coord & replace η by ξ

$$\Rightarrow \lambda(\xi+\pi) = -\lambda(\xi)$$

$\Leftarrow) \quad \lambda(\xi) = e^{i\xi} s(2\xi) \quad ; \quad s \in L^2_{2\pi}$

<u>why?</u> $\lambda(\xi+\pi) = -\lambda(\xi) \longleftrightarrow \lambda(\xi) = \sum_{k \in \mathbb{Z}} c_{2k+1} \, e^{i(2k+1)\xi}$

$$W_{-1} = \left\{ f \in L^2 : \hat{f}(\xi) = e^{i\xi} s(2\xi) \overline{m_0(\xi+\pi)} \hat{\varphi}(\xi) \quad \right\}$$

$$s \in L^2_{2\pi}$$

$$W_0 = \left\{ f \in L^2 : \hat{f}(2\xi) = e^{i\xi} s(2\xi) \overline{m_0(\xi+\pi)} \hat{\varphi}(\xi) \right\}$$

<u>Thm</u> ψ is wavelet $\Leftrightarrow$ $\hat{\psi}(2\xi) = e^{i\xi} \nu(2\xi) \overline{m_0(\xi+\pi)} \hat{\varphi}(\xi)$

$$\nu \in L^2_{2\pi}, \quad |\nu| = 1$$