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$$W_{-1} = \{ f \in L^2 : \hat{f}(\xi) = e^{i\xi} s(2\xi) \overline{m_0(\xi+\pi)} \hat{\varphi}(\xi), \ s \in L^2_{2\pi} \}$$

$$W_0 = \{ f \in L^2 : \hat{f}(2\xi) = e^{i\xi} s(2\xi) \overline{m_0(\xi+\pi)} \hat{\varphi}(\xi), \ s \in L^2_{2\pi} \}$$

Thm: $\psi \in W_0$ is a wavelet iff $\hat{\psi}(2\xi) = e^{i\xi} \nu(2\xi) \overline{m_0(\xi+\pi)} \hat{\varphi}(\xi)$

where $\nu(\xi) \in L^2_{2\pi}$ $|\nu(\xi)| = 1$ a.e.

Pf: Translations of ψ are ONS iff

$$\sum_{n\in\mathbb{Z}} |\hat{\psi}(\xi + 2n\pi)|^2 \equiv 1/2\pi$$

and
$$\sum_{n\in\mathbb{Z}} |\hat{\psi}(\xi+2n\pi)|^2 = \sum_{n\in\mathbb{Z}} |m_0(\tfrac{\xi}{2} + \pi n + \pi)|^2 |\hat{\varphi}(\tfrac{\xi}{2} + \pi n)|^2$$

$$= \sum_{\text{even}} + \sum_{\text{odd}} = \underbrace{\left(|m_0(\tfrac{\xi}{2} + \pi)|^2 + |m_0(\tfrac{\xi}{2})|^2\right)}_{\equiv 1} \frac{1}{2\pi}$$

Why do translates of ψ span all of W_0?

$$W_0 = \{ f\in L^2 : \hat{f}(\xi) = e^{i\xi/2} s(\xi) \overline{m_0(\xi/2+\pi)} \hat{\varphi}(\xi/2), \ s \in L^2_{2\pi} \}$$

and $\hat{\psi}(\xi) = e^{i\xi/2} \nu(\xi) \overline{m_0(\xi/2+\pi)} \hat{\varphi}(\xi/2)$

so $\widehat{\sum c_k \psi_k} = \hat{\psi}(\xi) \sum c_k e^{-ik\xi}$

Ex $\nu \equiv 1$ $\hat{\psi}(\xi) = \left(\sum_{k\in\mathbb{Z}} (-1)^k \bar{\alpha}_k e^{-i(k-1)\xi/2} \right) \hat{\varphi}(\xi/2) \Rightarrow \psi(x) = 2 \sum_{k\in\mathbb{Z}} (-1)^k \bar{\alpha}_k \varphi(2x - (k-1))$

Recall $\frac{1}{2} \varphi(\frac{1}{2}x) = \sum \alpha_k \varphi(x+k)$

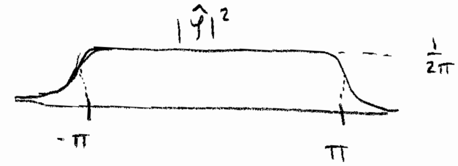
$$\alpha_k = \frac{1}{2} \int \varphi(\tfrac{x}{2}) \overline{\varphi(x+k)} \, dx$$

Exercise: If $\varphi = \chi_{[0,1)}$ then ψ is Haar wavelet.

Examples:

Lemarié - Meyer Wavelet:

Consider $\hat{\varphi}$:

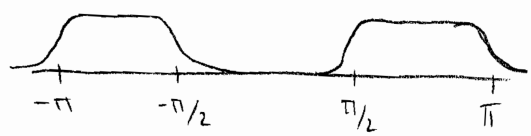


$$\hat{\varphi} \in C^\infty \qquad \text{even}$$

$$|\hat{\varphi}(\pi+s)|^2 + |\hat{\varphi}(\pi-s)|^2 \equiv \frac{1}{2\pi}$$

Then $\sum |\hat{\varphi}(\xi+2\pi n)|^2 \equiv \frac{1}{2\pi}$

$\dfrac{\hat{\psi}(2\xi)}{e^{i\xi}}$:



Thm: There are no wavelets in C_0^∞

Franklin Wavelets: use piecewise linear functions

ψ "roughly" supported around $[0,1]$

Wavelets in $\mathbb{R}^2$ (or $\mathbb{R}^n$)

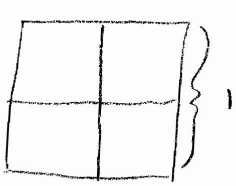
$\psi_k^n(x) \cdot \psi_j^m(y)$ Some of these will be supported on narrow rectangles

We want squares

V_0 - functions constant on dyadic squares

V_n - constant on dyadic squares of size 2^{-n}

$W_0 = V_1 \ominus V_0$ constant on dyadic squares of size $1/2$ and orthogonal to 1 on each dyadic square of size 1.



Basis: