

from the solvability of the Dirichlet problem in the case that ∂D is smooth. To prove it in the general case, use Exercise 4.

5. The Riemann Mapping Theorem Revisited

The Riemann mapping theory can be proved by solving a Dirichlet problem. This was Riemann's original strategy, though he could not solve the Dirichlet problem in the full generality of simply connected domains. We will give a proof along the lines envisioned by Riemann.

We say that a subset E of the extended complex plane \mathbb{C}^* is a **continuum** if it is compact and connected and has more than one point. If E is a continuum, then $\mathbb{C}^* \setminus E$ is simply connected. Conversely, if E is a compact set such that $\mathbb{C}^* \setminus E$ is simply connected, and if E has more than one point, then E is a continuum. (See Section VIII.8.) The following criterion allows us to solve the Dirichlet problem for any bounded simply connected domain.

Lemma. *Let D be a domain, and let $\zeta_0 \in \partial D$. If ζ_0 lies on a continuum in $\mathbb{C} \setminus D$, then ζ_0 is a regular boundary point of D .*

We construct a subharmonic barrier at ζ_0 . Let E be the continuum in $\mathbb{C} \setminus D$ containing ζ_0 , and let ζ_1 be any other point of E . Since $\mathbb{C}^* \setminus E$ is simply connected, $\log((z - \zeta_0)/(z - \zeta_1))$ has an analytic branch on $\mathbb{C} \setminus E$; call it $f(z)$. The set $D_0 = \{|z - \zeta_0|/|z - \zeta_1| < 1/e\}$ is a disk containing ζ_0 , and the values of $w = f(z)$ on D_0 lie in the half-plane $\{\operatorname{Re} w < -1\}$. Thus the values of $1/f(z)$ on $D \cap D_0$ lie in the disk $\{|w + \frac{1}{2}| < \frac{1}{2}\}$ in the left half-plane. Since $\operatorname{Re} f(z) \rightarrow -\infty$ if and only if $z \rightarrow \zeta_0$, $1/f(z) \rightarrow 0$ for $z \in D \cap D_0$ if and only if $z \rightarrow \zeta_0$. Hence the restriction of $\operatorname{Re}(1/f(z))$ to $D \cap D_0$ is a harmonic barrier at ζ_0 .

Corollary. *Let D be a bounded domain. If $\mathbb{C}^* \setminus D$ consists of finitely many continua, then every point of ∂D is a regular boundary point.*

Now we can prove the Riemann mapping theorem. Suppose D is a simply connected domain and that $\mathbb{C}^* \setminus D$ has at least two points. As in Section XI.3, we can use the logarithm function (or the square root function) and a fractional linear transformation to map D conformally onto a bounded domain. Indeed, let $\zeta_0, \zeta_1 \in \partial D$, and let $f(z)$ be an analytic branch of $\log((z - \zeta_0)/(z - \zeta_1))$ in D . If $w_0 = f(z_0)$ for some $z_0 \in D$, then the image $f(D)$ covers a disk D_0 centered at w_0 . Consequently, $f(z)$ cannot assume any of the values in $D_0 + 2\pi i$ on D , so that $1/(f(z) - w_0 - 2\pi i)$ maps D conformally onto a bounded domain.

So we assume that D is bounded, and we also assume for convenience that $0 \in D$. We solve the Dirichlet problem for the function $\log|\zeta|$ on ∂D ,