# 1

# THE DIRICHLET PROBLEM AND HARMONIC MEASURE

## 1.1. INTRODUCTION

Let $\Omega$ be a domain on the Riemann sphere and let $u$ be a continuous real-valued function on $\Gamma = \partial\Omega$. The Dirichlet problem is to find, if possible, a function $f$ which is continuous on $\Omega \cup \Gamma = \mathrm{CL}(\Omega)$ and which satisfies the following conditions:

1. $f$ is harmonic on $\Omega$; that is, $\Delta f = 0$ on $\Omega$.
2. $f = u$ on $\Gamma$.

There certainly are cases in which this problem is not solvable; for example, if $\Omega = \{z : 0 < |z| < 1\}$ and $u(0) = 1$, $u(e^{it}) = 0$, $0 \leqslant t \leqslant 2\pi$. However, there are a wide variety of domains for which it is solvable since there are quite reasonable conditions that are sufficient for solvability. The standard approach is by the method of Otto Perron and makes use of subharmonic functions. We begin in Section 2 with some material on the Poisson integral and proceed to the definition of subharmonic functions and some of their basic properties in Section 3; in Section 4, we use subharmonic functions to "solve" the Dirichlet problem. Related matters, including harmonic measure, the Green's function, and logarithmic capacity are covered in Sections 5, 6, and 7.

## 1.2. THE POISSON FORMULA AND SOME PRELIMINARIES

It is worthwhile to begin by recalling the Poisson integral formula and some of its basic properties. Set

$$P(r, \theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}, \qquad 0 < r < 1, \qquad 0 \leqslant \theta \leqslant 2\pi \tag{2.1}$$

$P(r, \theta)$ is the Poisson kernel and it is a simple matter to verify that

$$P(r,\theta) = \operatorname{Re}\left(\frac{1+z}{1-z}\right), \qquad z = re^{i\theta} \tag{2.2a}$$

$$P(r,\theta) > 0 \tag{2.2b}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta)\,d\theta = 1, \qquad 0 < r < 1 \tag{2.2c}$$

$$\text{for } \delta > 0, \ \lim_{r\to 1} \max\{P(r,\theta) : \delta \leqslant |\theta| \leqslant \pi\} = 0 \tag{2.2d}$$

Now let $u$ be a continuous function on the unit circle $\mathbf{T}$, $\mathbf{T} = \{e^{i\theta} : -\pi \leqslant \theta \leqslant \pi\}$, and set

$$P_u(re^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r, t-\theta)u(e^{i\theta})\,d\theta$$

The function $P_u$ is a harmonic function of $z = re^{it}$ as (2.2a) shows. The significant thing about $P_u$ is what $P_u(z)$ does as $z$ tends to a point of $\mathbf{T}$.

**Theorem 2.1.** *$P_u(z) \to u(\lambda)$ as $z \to \lambda$, $\lambda \in \mathbf{T}$; that is, $P_u$ is continuous on $\Delta \cup \mathbf{T}$ and coincides on $\mathbf{T}$ with $u$.*

***Proof.*** The proof is the standard application of an approximate identity argument and makes use of (2.2b), (2.2c), and (2.2d). Let $\lambda = e^{i\psi}$. Then

$$P_u(re^{it}) - u(e^{i\psi}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r, t-\theta)\left[u(e^{i\theta}) - u(e^{i\psi})\right] d\theta$$

$$= \int_{|\theta-\psi|<\delta} + \int_{|\theta-\psi|\geqslant\delta}$$

Given $\varepsilon > 0$, choose $\delta > 0$ so that whenever $|\theta - \psi| < \delta$ it follows that $|u(e^{i\theta}) - u(e^{i\psi})| < \varepsilon/2$; from now on we view $t$ as being restricted by $|t - \psi| < \delta/2$. Next, for this $\delta$, let $r_0$ be chosen so that whenever $r_0 \leqslant r < 1$ it follows that

$$\max\left\{P(r,s) : \frac{\delta}{2} \leqslant |s| \leqslant \pi\right\} < \frac{\varepsilon}{4M} \tag{2.3}$$

where $M = \max\{|u(\theta)| : |\theta| \leqslant \pi\}$. In the preceding identity for $P_u(re^{it}) - u(e^{i\psi})$ we estimate the first integral by $\varepsilon/2$ since (2.2b) and (2.2c) hold. We estimate the second integral by $2M \max\{P(r,s) : \delta/2 \leqslant s \leqslant \pi\} \leqslant \varepsilon/2$ because of (2.2b), (2.2d) and (2.3). Thus, the distance from $P_u(re^{it})$ to $u(e^{i\psi})$ is at most $\varepsilon$ when $|\psi - t| < \delta/2$ and $r \geqslant r_0$, which is precisely what was to be shown.

**Definition.** Let $\mu$ be a measure on $\mathbf{T}$ and set

$$P_\mu(re^{it}) = \int_{\mathbf{T}} P(r, t-\theta)\, d\mu(\theta) \tag{2.4}$$

The function $P_\mu$, which is harmonic in $\Delta$, behaves relatively well at the unit circle $\mathbf{T}$.

**Theorem 2.2.** *Let $d\mu = v\,d\theta + d\alpha$ be the Lebesgue decomposition of $\mu$ where $v \in L^1(\mathbf{T}, d\theta)$ and $d\alpha$ is singular with respect to $d\theta$. Then*

$$\lim_{r\to 1} P_\mu(re^{it}) = 2\pi v(t) \quad \text{a.e. } dt \tag{2.5}$$

***Proof.*** This proof is like that of Theorem 2.1 but, since we are only looking at radial limits, it is somewhat simpler. The measure $d\mu$ is given by the function $\mu$ of bounded variation on $[-\pi, \pi]$ and (2.5) will follow if we show that

$$\lim_{r\to 1} P_\mu(re^{i\theta_0}) = 2\pi\mu'(\theta_0) \tag{2.6}$$

at each point $\theta_0$ at which $\mu$ is differentiable. Note that if $d\mu = 1/(2\pi)\, d\theta$, then $P_\mu \equiv 1$ and so (2.6) holds. Hence, we may subtract a constant multiple of $1/(2\pi)\, d\theta$ from $d\mu$ and so assume that $\int d\mu = 0$; that is, $\mu(-\pi) = \mu(\pi)$.

One integration by parts leads to

$$P_\mu(re^{i\theta}) = \int_{-\pi}^{\pi} P'(r, \theta - t)\mu(t)\, dt$$

Now $P'(r, \psi) = [2r(1-r^2)\sin\psi]/[(1 - 2r\cos\psi + r^2)^2]$ and is an odd function of $\psi$. Thus, we obtain

$$\begin{aligned} P_\mu(re^{i\theta}) &= \int_{-\pi}^{\pi} P'(r, \theta - t)\mu(t)\, dt \\ &= \int_{-\pi}^{0} + \int_{0}^{\pi} \\ &= \int_{0}^{\pi} P_r'(t)[\mu(\theta - t) - \mu(\theta + t)]\, dt \\ &= \int_{-\pi}^{\pi} [-\sin t] P_r'(t) \left[\frac{\mu(\theta + t) - \mu(\theta - t)}{2\sin t}\right] dt \end{aligned}$$

The functions

$$K(r, t) = \frac{-1}{2\pi r}(\sin t) P_r'(t), \qquad 0 < r < 1$$

form an approximate identity; that is, (2.2b)–(2.2d) hold with $K(r, t)$ in place of $P(r, t)$. The function

$$F(t) = \frac{\mu(\theta_0 + t) - \mu(\theta_0 - t)}{2 \sin t}$$

is continuous at $t = 0$ with value $F(0) = \mu'(\theta_0)$. Hence

$$\lim_{r \to 1} 2\pi r \int_{-\pi}^{\pi} K(r, t) F(t)\, dt = 2\pi F(0)$$

$$= 2\pi\mu'(\theta_0)$$

which is the desired conclusion.

**Proposition 2.3.** A harmonic function $u$ in $\Delta$ has the representation

$$u(re^{i\theta}) = \int_{\mathbf{T}} P(r, \theta - t)\, d\mu(t) \tag{2.7}$$

for some measure $\mu$ on $\mathbf{T}$ if and only if

$$\sup\left\{ \int_{-\pi}^{\pi} |u(re^{i\theta})|\, d\theta : r < 1 \right\} \text{ is finite} \tag{2.8}$$

If (2.7) holds, then $\mu$ is uniquely determined.

***Proof.*** Suppose (2.7) holds. Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta})|\, d\theta \leqslant \frac{1}{2\pi} \int_{\mathbf{T}} \int_{-\pi}^{\pi} P(r, \theta - t)\, d\theta\, d|\mu|(t)$$

$$= \int_{\mathbf{T}} d|\mu(t)| = \|\mu\|, \text{ the total variation of } \mu$$

Here we have made use of (2.2b) and (2.2c). Conversely, suppose (2.8) holds. Let $\mu_\rho$ be the measure on $\mathbf{T}$ given by

$$d\mu_\rho(t) = \frac{1}{2\pi} u(\rho e^{it})\, dt, \qquad 0 < \rho < 1$$

so that (2.8) says exactly that the total variation of $\mu_\rho$ is uniformly bounded for $0 < \rho < 1$. There thus is a measure $\mu$ on $\mathbf{T}$ which is a weak-* cluster point of

$\{\mu_\rho\}$. Hence,

$$\begin{aligned} u(re^{i\theta}) &= \lim_{\rho\to 1} u(\rho r e^{i\theta}) \\ &= \lim_{\rho\to 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta - t) u(\rho e^{it})\, dt \\ &= \lim_{\rho\to 1} \int_{\mathbf{T}} P(r,\theta - t)\, d\mu_\rho(t) \\ &= \int_{\mathbf{T}} P(r,\theta - t)\, d\mu(t) \end{aligned}$$

As for uniqueness, suppose $\mu_1$ also satisfies (2.7). The difference $\mu - \mu_1$ then annihilates all $P(r, \theta - t)$ and we must show that this implies $\mu - \mu_1$ is zero. By taking real and imaginary parts we need only show that if $\nu$ is a real measure with

$$0 = \int_{\mathbf{T}} P(r,\theta - t)\, d\nu(t), \qquad 0 < r < 1, \qquad \theta \in [0, 2\pi] \tag{2.9}$$

then $\nu = 0$. From (2.9) and (2.2a) we have

$$0 = \operatorname{Re}\int_{\mathbf{T}} \frac{e^{it} + z}{e^{it} - z}\, d\nu(t), \qquad |z| < 1$$

so that the analytic function

$$\int_{\mathbf{T}} \frac{e^{it} + z}{e^{it} - z}\, d\nu(t) = h(z)$$

is identically constant and hence 0 since $h(0) = 0$. However,

$$h(z) = \int_{\mathbf{T}} d\nu + 2\sum_1^\infty z^n \left\{ \int_{\mathbf{T}} e^{-int}\, d\nu(t) \right\}$$

Hence,

$$0 = \int_{\mathbf{T}} e^{-int}\, d\nu(t), \qquad n = 0, 1, 2, \ldots$$

Since $\nu$ is real, this implies $\nu$ is the zero measure.

**Corollary 2.4.** *If $u$ is a positive harmonic function on $\Delta$ then there is a unique non-negative measure $\mu$ on* **T** *with*

$$u(re^{i\theta}) = \int_{\mathbf{T}} P(r, \theta - t)\, d\mu(t) \tag{2.10}$$

***Proof.*** Since $u$ is positive it satisfies (2.8):

$$u(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(re^{i\theta})\, d\theta$$

Thus, (2.10) holds for some measure $\mu$; note that Proposition 2.3 actually yielded the information that

$$\|\mu\| \leqslant \sup\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(re^{i\theta})|\, d\theta : 0 < r < 1\right\}$$

In this case, this supremum is $u(0)$; hence

$$\|\mu\| \leqslant u(0) = \int_{\mathbf{T}} d\mu$$

so that $\mu$ is non-negative.

## 1.3. SUBHARMONIC FUNCTIONS

**Definition.** A function $u(z)$ defined for $z$ in a domain $\Omega$ on the sphere is *subharmonic* on $\Omega$ if it satisfies the following conditions:

$$-\infty \leqslant u(z) < \infty, \qquad z \in \Omega \tag{3.1a}$$

$u$ is upper semicontinuous on $\Omega$; that is,

$$u(a) \geqslant \limsup\{u(z) : z \to a\} \text{ for all } a \in \Omega \tag{3.1b}$$

if the closed disc $\{z : |z - p| \leqslant r\}$ lies in $\Omega$, then

$$u(p) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} u(p + re^{it})\, dt \tag{3.1c}$$

Evidently every real-valued harmonic function on $\Omega$ is subharmonic and if both $u$ and $-u$ are subharmonic then $u$ is harmonic, since in this case $u$ will be continuous and equality will hold in (3.1c). It is also evident that the sum and the maximum of two subharmonic functions are also subharmonic, as is a positive multiple of a subharmonic function. We gather some simple facts about subharmonic functions in the next several propositions.

**Proposition 3.1.** Let $u$ be subharmonic on $\Omega$ and let $\phi$ be a monotonically increasing convex function on $\mathbb{R}$. Then $\phi(u(z))$ is subharmonic on $\Omega$.

***Proof.*** If we set $v(z) = \phi(u(z))$ then it is apparent that $v$ satisfies (3.1a) and (3.1b). Further, we have

$$\begin{aligned} v(p) = \phi(u(p)) &\leqslant \phi\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} u(p + re^{it})\,dt\right) \\ &\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi\big(u(p + re^{it})\big)\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} v(p + re^{it})\,dt \end{aligned}$$

since $\phi$ is both increasing and convex.

EXAMPLE.

Let $f$ be holomorphic on $\Omega$. Then both $\log|f|$ and $|f|^q$, $0 < q < \infty$, are subharmonic on $\Omega$.

This is reasonably straightforward. Clearly (3.1a) and (3.1b) are satisfied. Further, (3.1c) is direct for $|f|^q$ if $1 \leqslant q < \infty$ since

$$\begin{aligned} |f(p)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(p + re^{it})\,dt\right| \\ &\leqslant \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(p + re^{it})|^q\,dt\right)^{1/q} \end{aligned}$$

by Hölder's inequality. However, this argument won't work for $0 < q < 1$, so we first show $\log|f|$ is subharmonic and then apply Proposition 3.1 with $\phi(t) = e^{qt}$.

Suppose that $\{z : |z - p| \leqslant r\}$ is in $\Omega$; we assume first that $f \neq 0$ on $|z - p| = r$. Let $z_1, \ldots, z_N$ be the zeros of $f$ in $|z - p| < r$, and put

$$g(z) = f(z)\prod_{1}^{N} \frac{r^2 - \overline{(z_j - p)}(z - p)}{r(z - z_j)}, \qquad |z - p| \leqslant r$$

Then $g$ is holomorphic on $\{|z - p| \leqslant r\}$, $|g(z)| = |f(z)|$ if $|z - p| = r$, and $|g(z)| > |f(z)|$ if $|z - p| < r$. Further, $g$ is zero-free in $|z - p| \leqslant r$ so that $\log|g|$ is harmonic there. Thus,

$$\begin{aligned} \log|f(p)| < \log|g(p)| &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|g(p + re^{it})|\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(p + re^{it})|\,dt \end{aligned}$$

Finally, if $f$ does vanish on $|z-p|=r$ choose $r_n \uparrow r$ so that $f \neq 0$ on $|z-p|=r_n$. Thus,

$$\log|f(p)| \leqslant \limsup_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(p+r_n e^{it})|\,dt$$

$$\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \limsup_{n\to\infty} \log|f(p+r_n e^{it})|$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(p+re^{it})|\,dt$$

which establishes the desired inequality.

**Proposition 3.2.** Let $\{u_n\}$ be a sequence of subharmonic functions on $\Omega$ and suppose that $u_1(z) \geqslant u_2(z) \geqslant \cdots$, $z \in \Omega$ and $\lim_{n\to\infty} u_n(z_0) = L > -\infty$ for some $z_0 \in \Omega$. Then $u(z) = \lim_{n\to\infty} u_n(z)$ is subharmonic on $\Omega$.

***Proof.*** Suppose that $p \in \Omega$ is a point at which $-\infty < u(p)$. If the disc $\{z : |z-p| \leqslant r\}$ lies in $\Omega$, then

$$u(p) \leqslant u_n(p) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} u_n(p+re^{it})\,dt$$

But the latter integrals converge to the integral of $u$ by the monotone convergence theorem. Hence, $u$ satisfies (3.1c) at $p$. Moreover, $u(p+re^{it}) > -\infty$ for almost all points $(dt)$ on the circle $|z-p|=r$.

Next let $a$ be any point of $\Omega$ at which $u(a) > -\infty$. Then, given $\varepsilon > 0$, there is a large $n$ for which $u(a) > u_n(a) - \varepsilon$. Thus,

$$u(a) > u_n(a) - \varepsilon \geqslant \limsup_{z\to a} u_n(z) - \varepsilon$$

$$\geqslant \limsup_{z\to a} u(z) - \varepsilon$$

and so $u$ is upper semicontinuous at $a$. If $u(a) = -\infty$, then given a large positive number $M$, we know that $u_n(a) < -M$ for all large $n$. Hence,

$$\limsup_{z\to a} u(z) \leqslant \limsup_{z\to a} u_n(z) \leqslant -M$$

so that $\limsup_{z\to a} u(z) = -\infty$, in the case $u(a) = -\infty$.

**Proposition 3.3.** Suppose $u \in C^2(\Omega)$. If $\Delta u \geqslant 0$ in $\Omega$, then $u$ is subharmonic on $\Omega$.

***Proof.*** Suppose the disc $D = \{z : |z - p| \leqslant r\}$ lies in $\Omega$. Green's theorem gives

$$\iint_D \Delta u \, dx \, dy = \int_{-\pi}^{\pi} r \frac{\partial}{\partial t} u(p + te^{i\theta}) \, d\theta$$

$$= r \frac{d}{dt} \left( \int_{-\pi}^{\pi} u(p + te^{i\theta}) \, d\theta \right) \bigg|_{t=r}$$

However, the first integral is non-negative so that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} u(p + te^{i\theta}) \, d\theta$$

is an increasing function of $t$. As $t \to 0$, this integral converges to $u(p)$, by continuity. Hence, $u$ satisfies (3.1c).

We need a simple fact about upper semicontinuous functions which we isolate here; its proof is left as an exercise.

**Lemma 3.4.** Let $\mathbf{K}$ be a compact set and let $u$ be a function on $\mathbf{K}$ with values in $[-\infty, \infty)$. Then $u$ is upper semicontinuous if and only if there is a sequence $\{f_n\}$ of continuous functions on $\mathbf{K}$ with $f_1 \geqslant f_2 \geqslant \cdots$ and $\lim f_n(z) = u(z)$, $z \in \mathbf{K}$.

The most important fact about subharmonic functions and the origin of their name is contained in the next result.

**Theorem 3.5.** *An upper semicontinuous function $u$ on $\Omega$ with values in $[-\infty, \infty)$ is subharmonic if and only if, whenever $\mathbf{K}$ is a compact subset of $\Omega$ and $h$ is a function continuous on $\mathbf{K}$ and harmonic on* INT $\mathbf{K}$ *with $h \geqslant u$ on $\partial\mathbf{K}$, then $h \geqslant u$ on* INT $\mathbf{K}$ *as well.*

***Proof.*** One direction of the theorem is straightforward. Suppose $u$ has the property described in the second half of the theorem. If $D = \{z : |z - p| \leqslant r\}$ lies in $\Omega$, then let $\{f_n\}$ be a sequence of continuous functions on the compact set $\gamma = \{z : |z - p| = r\}$ which decrease to $u$. Again denote by $f_n$ the function continuous on $\{z : |z - p| \leqslant r\}$, harmonic on $\{z : |z - p| < r\}$ which equals $f_n$ on $\gamma$. Then, we have $f_n \geqslant u$ on $\gamma$ so that $f_n \geqslant u$ inside $\gamma$ as well, by the hypothesis. Thus,

$$u(p) \leqslant f_n(p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_n(p + re^{it}) \, dt$$

But the integrals converge to $\frac{1}{2\pi} \int_{-\pi}^{\pi} u(p + re^{it}) \, dt$ by the monotone convergence theorem. Thus, $u$ satisfies (3.1c).

Conversely, let $u$ be subharmonic in $\Omega$, let $\mathbf{K}$ be a compact set in $\Omega$ and let $h$ be continuous on $\mathbf{K}$, harmonic on $\text{INT}\,\mathbf{K}$, and $h \geqslant u$ on $\partial\mathbf{K}$. Set $v = u - h$ so that we wish to show $v \leqslant 0$ on $\text{INT}\,\mathbf{K}$. If not, let

$$m = \text{lub}\{v(z) : z \in \mathbf{K}\}$$

and

$$\mathbf{E} = \{z \in \mathbf{K} : v(z) = m\}$$

Since $v$ is upper semicontinuous, $\mathbf{E}$ is nonempty and closed. Since $v \leqslant 0$ on $\partial\mathbf{K}$, we also know that $\mathbf{E}$ is a subset of $\text{INT}\,\mathbf{K}$. Let $p$ be a point in $\partial\mathbf{E}$ and choose $r > 0$ so small that $\{z : |z - p| \leqslant r\}$ lies in $\text{INT}\,\mathbf{K}$. Then on some arc of the circle $\gamma = \{z : |z - p| = r\}$ we have $v \leqslant m - \delta$, $\delta > 0$, and everywhere on $\gamma$ we have $v \leqslant m$. But then we obtain the contradiction

$$m = v(p) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} v(p + re^{it})\,dt < m$$

This proves that $v \leqslant 0$ in $\text{INT}\,\mathbf{K}$, as desired.

**Proposition 3.6.** If $u \in C^2(\Omega)$, then $u$ is subharmonic if and only if $\Delta u \geqslant 0$ on $\Omega$.

***Proof.*** Suppose $u$ is subharmonic and $\{z : |z - p| \leqslant r\}$ lies in $\Omega$. If $0 < r_1 < r_2 \leqslant r$, let $v$ be the harmonic function on $|z - p| < r_2$ which agrees with $u$ on $|z - p| = r_2$. Then $u \leqslant v$ in $|z - p| < r_2$ so that

$$\begin{aligned}\int_{-\pi}^{\pi} u(p + r_1e^{i\theta})\,d\theta &\leqslant \int_{-\pi}^{\pi} v(p + r_1e^{i\theta})\,d\theta = 2\pi v(p)\\ &= \int_{-\pi}^{\pi} v(p + r_2e^{i\theta})\,d\theta\\ &= \int_{-\pi}^{\pi} u(p + r_2e^{i\theta})\,d\theta\end{aligned}$$

Thus, $\int_{-\pi}^{\pi} u(p + te^{i\theta})\,d\theta$ is an increasing function of $t$ for $0 < t \leqslant r$ and so

$$\begin{aligned}0 \leqslant r\frac{d}{dt}\int_{-\pi}^{\pi} u(p + te^{i\theta})\,d\theta &= \int_{-\pi}^{\pi} r\left(\frac{\partial}{\partial t}u(p + te^{i\theta})\right)d\theta\\ &= \iint_{|z-p|\leqslant r} \Delta u\,dx\,dy\end{aligned}$$

by Green's theorem. This clearly implies $\Delta u \geqslant 0$ in $\Omega$.

There is a "maximum modulus" theorem for subharmonic functions.

**Proposition 3.7.** Suppose there is a number $M < \infty$ such that

$$\limsup\{u(z) : z \to \zeta\} \leqslant M \quad \text{for all } \zeta \in \partial\Omega \tag{3.2}$$

Then $u(z) \leqslant M$ for all $z \in \Omega$. If $u(z_0) = M$ for some $z_0 \in \Omega$, then $u \equiv M$ in $\Omega$.

The proof is virtually the same as that of Theorem 3.5 and we leave it as an exercise.

As a finale to this section we note that subharmonicity is preserved by conformal maps, a result we will have need of later. Specifically, let $\phi$ be a one-to-one holomorphic mapping of a domain $\Omega$ onto a domain $\Omega_1$ and suppose $u_1$ is subharmonic in $\Omega_1$. Then $u(z) = u_1(\phi(z))$ is subharmonic in $\Omega$, as is easily verified by use of Theorem 3.5.

## 1.4. SOLUTION OF THE DIRICHLET PROBLEM

The fundamental result needed to attack the Dirichlet problem is this.

**Proposition 4.1.** Let $\mathfrak{F}$ be a family of subharmonic functions satisfying these two conditions:

$$\text{whenever } u, v \in \mathfrak{F}, \text{ then } \max(u, v) \text{ also lies in } \mathfrak{F} \tag{4.1}$$

$$\text{if} \{z : |z - p| \leqslant r\} \subset \Omega \text{ and if } u \in \mathfrak{F}, \text{ then the function} \tag{4.2}$$

$$s(u, z) = \begin{cases} u(z) & \text{if } |z - p| \geqslant r, \qquad z \in \Omega \\ P_u(z) & \text{if } |z - p| < r \end{cases}$$

also lies in $\mathfrak{F}$.

Set

$$v(z) = \sup\{u(z) : u \in \mathfrak{F}\} \tag{4.3}$$

Then either $v \equiv +\infty$ in $\Omega$ or $v$ is harmonic in $\Omega$.

***Proof.*** Recall that $P_u(z)$ is the Poisson extension of $u$ to the disc which, in this case is the disc $|z - p| < r$.

Suppose first that there is some point $z_0 \in \Omega$ at which $v(z_0) = \infty$. Choose elements $u_1, u_2, \ldots$ of $\mathfrak{F}$ so that $\{u_n(z_0)\}$ increases to $\infty$. By (4.1) we may replace $u_n$ by $v_n = \sup\{u_1, \ldots, u_n\}$ and still have $v_n \in \mathfrak{F}$. However, $v_1 \leqslant v_2 \leqslant \cdots$ on all of $\Omega$ and $v_n(z_0) \to \infty$. Further, in the disc $D = \{|z - z_0| \leqslant r\}$ we can replace $v_n$ by $P_{v_n}$ and the resulting function $s(v_n, z)$ is again in $\mathfrak{F}$ by (4.2).

But

$$s(v_n, z_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v_n(z_0 + re^{it})\, dt$$

$$\geq v_n(z_0)$$

so that $s(v_n, z_0) \to \infty$ as $n \to \infty$ and so $s(v_n, z) \to \infty$ for all $z$, $|z - z_0| < r$. Hence, $v(z) = \infty$ if $|z - z_0| < r$. This implies that the set $\Omega' = \{z \in \Omega : v(z) = \infty\}$ is open. A moment's thought shows that the same argument implies that $\Omega'$ is also closed. Hence, $\Omega' = \Omega$ and $v = \infty$ throughout $\Omega$.

Suppose now that $v$ is finite at all points of $\Omega$; we shall show that $v$ is harmonic in $\Omega$. Let $a$ be a point of $\Omega$ and let $D$ be a disc centered at $a$ whose closure lies in $\Omega$. Let $\{u_n\}$ be a sequence of elements of $\mathfrak{F}$ with $u_n(a) \to v(a)$. By (4.1) we may replace $u_n$ by $\max\{u_1, \dots, u_n\}$ and still remain in $\mathfrak{F}$, so there is no loss in assuming $u_1 \leq u_2 \leq \cdots$. Using the disc $D$, we may employ (4.2) and assume as well that each $u_n$ is harmonic in $D$. Hence, the sequence $\{u_n\}$ increases on $\Omega$ to a function $U$ which is harmonic in $D$ and equals $v(a)$ at $a$. Now take $b$ to be any point of $D$, $b \neq a$. We may apply the same sort of reasoning to find a sequence $\{w_n\}$ of elements of $\mathfrak{F}$ with $w_1 \leq w_2 \leq \cdots$ on $\Omega$ and $w_n(b) \to v(b)$. Let $r_n$ be the function in $\mathfrak{F}$ which is harmonic in $D$ and which equals $\max\{u_n, w_n\}$ on $\partial D$. Then $w_n(b) \leq r_n(b)$ since $w_n$ is subharmonic and likewise $u_n(a) \leq r_n(a)$. Further, $\{r_n\}$ increases to a function $R$ which is necessarily harmonic in $D$ with $U(z) \leq R(z)$, $z \in D$. But we also know that $U(a) = v(a) \leq R(a)$ and $U(b) \leq v(b) \leq R(b)$. Hence $R - U$ is a non-negative harmonic function which vanishes at $a$ and so is identically zero. Consequently, $R(b) = U(b)$ as well so that $v(b) = U(b)$. Thus, $v \equiv U$ in $D$ and so $v$ is harmonic in $\Omega$.

Although Theorem 4.1 produces a harmonic function in $\Omega$ it does not make any reference to behavior at $\partial\Omega$, which is the essential issue in the Dirichlet problem. This aspect is attacked by means of a barrier, which we now define.

**Definition.** Let $x \in \partial\Omega$. There is a barrier at $x$ if for each small $\delta > 0$ it is possible to find a function $b(z)$, which may depend on $\delta$, such that

$$-b \text{ is subharmonic in } \Omega \tag{4.4a}$$

$$b \geq 0 \text{ in } \Omega \tag{4.4b}$$

$$b(z) \geq 1 \text{ if } z \in \Omega \quad \text{and} \quad |z - x| \geq \delta \tag{4.4c}$$

$$b(z) \to 0 \text{ if } z \in \Omega \quad \text{and} \quad z \to x \tag{4.4d}$$

The notion of a barrier and Theorem 4.1 can now be combined to give a point-by-point solution to the Dirichlet problem.

Let $h$ be a bounded function on $\partial\Omega$ and let $\mathfrak{F}(h)$ consist of all subharmonic functions $u$ on $\Omega$ which satisfy

$$\limsup\{u(z) : z \in \Omega, z \to \zeta\} \leqslant h(\zeta), \qquad \text{all } \zeta \in \partial\Omega \tag{4.5}$$

Then set

$$v(z) = v_h(z) = \sup\{u(z) : u \in \mathfrak{F}(h)\} \tag{4.6}$$

**Theorem 4.2.** *The function $v$ given in* (4.6) *is harmonic on $\Omega$. Further, if $h$ is continuous at $x \in \partial\Omega$ and if there is a barrier at $x$, then*

$$\lim_{z \to x} v(z) = h(x) \tag{4.7}$$

**Corollary 4.3.** *If there is a barrier at each point of $\partial\Omega$, then the Dirichlet problem is solvable for $\Omega$.*

***Proof.*** We now proceed to the proof of Theorem 4.2. We know from Proposition 3.7 that each function in $\mathfrak{F}(h)$ is bounded above by $M = \sup\{h(\zeta) : \zeta \in \partial\Omega\}$ and so the function $v$ given by (4.6) is harmonic on $\Omega$.

Let $\varepsilon > 0$ be given; choose $\delta > 0$ so small that $|h(x) - h(y)| < \varepsilon/2$ if $y \in \partial\Omega$ and $|x - y| < \delta$. Let $b$ be the barrier for this $\delta$. Consider

$$s(z) = h(x) - \varepsilon - 2Mb(z), \qquad z \in \Omega$$

Suppose $y \in \partial\Omega$ and $|y - x| \leqslant \delta$; then

$$\limsup\{s(z): z \to y\} \leqslant h(x) - \varepsilon < h(y)$$

If $y \in \partial\Omega$ and $|y - x| > \delta$, then

$$\limsup\{s(z): z \to y\} \leqslant h(x) - 2M < h(y)$$

Thus, $s \in \mathfrak{F}(h)$ so that $v(z) \geqslant s(z)$ for all $z \in \Omega$. As a consequence, we have

$$\begin{aligned}\liminf\{v(z): z \to x\} &\geqslant \liminf\{s(z): z \to x\}\\ &\geqslant h(x) - \varepsilon\end{aligned}$$

Since $\varepsilon$ is arbitrary, we have

$$\liminf\{v(z): z \to x\} \geqslant h(x) \tag{4.8}$$

Consider next the family $\mathfrak{F}(-h)$ and put

$$w(z) = -\sup\{u(z) : u \in \mathfrak{F}(-h)\}$$

Then $w$ is harmonic in $\Omega$ and

$$\liminf\{-w(z): z \to x\} \geqslant -h(x)$$

or, equivalently,

$$\limsup\{w(z): z \to x\} \leqslant h(x) \tag{4.9}$$

Finally, if $u_1 \in \mathfrak{F}(h)$ and $u_2 \in \mathfrak{F}(-h)$, then $u_1 + u_2$ is subharmonic in $\Omega$ and

$$\begin{aligned}\limsup\{u_1(z) + u_2(z): z \to \zeta\} &\leqslant \limsup u_1 + \limsup u_2 \\ &\leqslant h(\zeta) + (-h(\zeta)) = 0\end{aligned}$$

so that $u_1 + u_2 \leqslant 0$ in $\Omega$. Hence, $v - w \leqslant 0$ in $\Omega$. This immediately gives the desired conclusion at $x$, for

$$\begin{aligned}h(x) &\geqslant \limsup\{w(z): z \to x\} \\ &\geqslant \limsup\{v(z): z \to x\} \\ &\geqslant \liminf\{v(z): z \to x\} \\ &\geqslant h(x)\end{aligned}$$

Thus, $\lim\{v(z): z \to x\} = h(x)$, as asserted.

Quite obviously now all that remains to be done to solve the Dirichlet problem is to give some sort of a condition that guarantees that there is a barrier; luckily there is a relatively good one that is strictly geometric. Recall that a continuum is a closed connected set consisting of more than one point.

**Theorem 4.4.** *Let $\Omega$ be a domain and let $x \in \partial\Omega$. If there is a continuum in the complement of $\Omega$ which contains $x$, then there is a barrier at $x$.*

***Proof.*** Let $x'$ be another point in the continuum. There is a linear fractional transformation which sends $x$ to $\infty$ and $x'$ to 0. We may thus restrict ourselves to the case when $x = \infty$ and the continuum $\mathbf{C}$ in the complement of $\Omega$ contains both 0 and $\infty$.

There is a single-valued branch of $\log z$ in the domain $\mathfrak{D} = \mathbf{C} \setminus \mathbf{C}$; note that $\Omega \subset \mathfrak{D}$. Let $\mathfrak{R}$ be the image of $\mathfrak{D}$ under $\log z$ so that $\mathfrak{R}$ is also a domain. There is no loss of generality in assuming that $\mathfrak{R}$ meets the imaginary axis (otherwise, just replace $\log z$ by $\log z - \alpha$ for some $\alpha \in \mathbb{R}$.) Let us write

$$\mathfrak{R} \cap \{it : t \in \mathbb{R}\} = \bigcup_{j=1}^{\infty} (i\alpha_j, i\beta_j)$$

where

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots \quad \text{and} \quad \sum_{j=1}^{\infty} (\beta_j - \alpha_j) \leqslant 2\pi \tag{4.10}$$

(there may, of course, be only a finite number of the intervals $(\alpha_j, \beta_j)$; this is not relevant.)

Define

$$h_j(z) = \arg\left(\frac{z - i\alpha_j}{z - i\beta_j}\right), \qquad \operatorname{Re} z > 0, \qquad j = 1, 2, \ldots$$

and then

$$h(z) = -\frac{1}{\pi} \sum_{j=1}^{\infty} h_j(z), \qquad \operatorname{Re} z > 0 \tag{4.11}$$

Then each $h_j$ is harmonic on $\operatorname{Re} z > 0$ and $h$ is the increasing limit of the partial sums of its series, so $h$ is also harmonic on $\operatorname{Re} z > 0$. Further, $-1 < h(z) < 0$.

If $x \in (\alpha_j, \beta_j)$ for some $j$ and if $\{z_m\}$ is a sequence in $\operatorname{Re} z > 0$ with $z_m \to ix$, then $h_j(z_m) \to \pi$ while $h_k(z_m) \to 0$ as $m \to \infty$, $k \neq j$. Hence, $h$ is continuous at $ix$ with $h(ix) = -1$. Finally, if $\operatorname{Re} z_m > 0$ and $|z_m| \to \infty$ then $h(z_m) \to 0$. Set

$$g(z) = \begin{cases} -1 & \text{if } \operatorname{Re} z \leqslant 0, \quad z \in \mathfrak{R} \\ h(z) & \text{if } \operatorname{Re} z > 0, \quad z \in \mathfrak{R} \end{cases} \tag{4.12}$$

Then $g$ is continuous in $\mathfrak{R}$, subharmonic on $\mathfrak{R}$, $-1 \leqslant g(z) \leqslant 0$, and $g(z) \to 0$ if $\operatorname{Re} z > 0$ and $|z| \to \infty$.

Now set

$$G(z) = g(\log z), \qquad z \in \mathfrak{D}$$

Then $G$ is subharmonic in $\mathfrak{D}$, $-1 \leqslant G \leqslant 0$, and $G(z) \to 0$ as $|z| \to \infty$. However, it may happen that $G \to 0$ at some finite boundary point. To compensate for this, let $\{t_n\}$ be real numbers increasing to $\infty$ so that the lines $\operatorname{Re} z = t_n$ all meet $\mathfrak{R}$. Let $g_n$ be the function corresponding to the line $\operatorname{Re} z = t_n$ and put

$$H(z) = \sum_{n=1}^{\infty} \frac{1}{2^n} g_n(\log z), \qquad z \in \mathfrak{D}$$

The series is uniformly convergent so $H$ is actually continuous on $\mathfrak{D}$, subharmonic there, and $-1 \leqslant H \leqslant 0$. Moreover,

$$\lim\{H(z): |z| \to \infty; z \in \mathcal{D}\} = 0$$

If $y$ is any finite point in $\partial\mathcal{D}$, then $\log y$ is a finite point in $\partial\mathcal{R}$ and so $g_n(\log y) = -1$ for all $n \geqslant n_0$. Hence

$$\limsup\{H(z) : z \in \mathcal{D}, z \to y\} < 0$$

Thus, if we are given a (large) M and we put

$$\rho = \sup_{|y| \leqslant M} \{\limsup\{H(z) : z \in D, z \to y\}\}$$

then $\rho < 0$ so that the function $\rho^{-1}H(z)$ is the desired function for the barrier.

**Corollary 4.5.** *If each component of $\partial\Omega$ is nontrivial, then the Dirichlet problem is solvable in $\Omega$.*

We should note that the existence of a barrier at each point of $\partial\Omega$ is not only sufficient to solve the Dirichlet problem in $\Omega$, it is also necessary. To see this let $x$ be a point in $\partial\Omega$ and let $u$ be a continuous function on $\partial\Omega$ with $u(x) = 0$ and $0 < u(y) \leqslant 1$ for all $y \in \partial\Omega$, $y \neq x$. If $h$ is the harmonic extension to $\Omega$ of $u$, then $\rho^{-1}h(z)$ is a barrier at $x$ for an appropriately small $\rho$.

## 1.5. THE GREEN'S FUNCTION OF A DOMAIN

Suppose that $\Omega$ is a domain on the sphere and that $p \in \Omega$. A function $g(z; p)$ is a Green's function for $\Omega$ with pole (or singularity) at $p$, $p \neq \infty$, if

$$g(z; p) \text{ is harmonic on } \Omega - \{p\} \tag{5.1a}$$

$$g(z; p) + \log|z - p| \text{ is harmonic near } p \tag{5.1b}$$

$$\lim\{g(z; p): z \to \zeta\} = 0 \quad \text{for all } \zeta \in \partial\Omega \tag{5.1c}$$

If $p = \infty$, then (5.1b) is modified to

$$g(z; \infty) - \log|z| \text{ is harmonic near } \infty \tag{5.1b$'$}$$

We begin by compiling a (short) list of some features of the Green's function.

**Proposition 5.1.** (a) The Green's function is unique, if it exists.

(b) If $\phi$ is a one-to-one holomorphic mapping of a domain $\Omega$ onto a domain $\Omega_1$ and if $\Omega_1$ has a Green's function $g_1(z; p_1)$ with singularity at $p_1 = \phi(p)$, then $g(z; p) = g_1(\phi(z); p_1)$ is the Green's function for $\Omega$ with singularity at $p$.

***Proof.*** (a) If $h(z;\, p)$ is another function on $\Omega$ with properties (5.1a), (5.1b), (5.1c), then the difference $g(z;\, p) - h(z;\, p)$ is harmonic on all of $\Omega$ and tends to zero at every point of $\partial\Omega$. Hence, it is identically zero.

(b) Clearly $g(z;\, p)$ has properties (5.1a) and (5.1c). To check (5.1b) let us assume that neither $p$ nor $p_1$ is the point at $\infty$. Then, near $p$,

$$g(z;\, p) + \log|z - p| = g_1(\phi(z);\, p_1) + \log|z - p|$$

$$= u(z) - \log|\phi(z) - \phi(p)| + \log|z - p|$$

where $u$ is harmonic near $p$. However, because $\phi'(p) \neq 0$ we have

$$\phi(z) - \phi(p) = (z - p)\psi(z)$$

where $\psi(z)$ does not vanish near $z = p$, and hence $\log|\psi|$ is harmonic near $p$. Thus,

$$g(z;\, p) + \log|z - p| = u(z) - \log|\psi(z)|$$

near $z = p$ and so $g$ satisfies (5.1b). If $p$ or $p_1$ is $\infty$, the proof is similar.

**Theorem 5.2.** *Let $\Omega$ be a domain for which the Dirichlet problem is solvable and let $p \in \Omega$. Then $\Omega$ has a Green's function with pole at $p$.*

***Proof.*** By an initial linear fractional transformation we may assume $p = \infty$ and that $\partial\Omega$ is compact. Let $h$ be the harmonic function on $\Omega$ whose values on $\partial\Omega$ are $-\log|z|$, and set

$$g(z;\, \infty) = h(z) + \log|z|, \qquad z \in \Omega$$

Then $g$ is harmonic on $\Omega \setminus \{\infty\}$, $g$ vanishes on $\partial\Omega$, and near $\infty$, $g(z;\, \infty) - \log|z| = h(z)$ so that $g$ satisfies (5.1b′).

It turns out that there is a "Green's function" for most every domain, whether or not the Dirichlet problem is solvable but it is necessary to relax condition (5.1c). The construction of the Green's function is by means of a "regular exhaustion" of $\Omega$, which we now define.

**Definition.** Let $\Omega$ be a domain. A regular exhaustion of $\Omega$ is a sequence $\{\Omega_n\}$ of subdomains of $\Omega$ satisfying

$$\mathrm{CL}(\Omega_n) \subset \Omega_{n+1}, \qquad n = 1, 2, \ldots \tag{5.2a}$$

$$\bigcup_{n=1}^{\infty} \Omega_n = \Omega \tag{5.2b}$$

$$\text{each component of } \partial\Omega_n \text{ is nontrivial} \tag{5.2c}$$

**Proposition 5.3.** Each domain has a regular exhaustion.

***Proof.*** There is no loss in assuming $\infty \in \Omega$ so that $\partial\Omega$ is compact. Cover $\partial\Omega$ by a finite number of open discs of radius $1/n$ and let $D_n$ be the complement of the union of the closures of these discs. Let $\Omega_n$ be the component of $D_n$ which contains $\infty$.

Let $\Omega$ be any proper subdomain of the Riemann sphere and let $p \in \Omega$. Let $\{\Omega_n\}$ be any regular exhaustion of $\Omega$; there is no loss in assuming that $p \neq \infty$ and that $p$ lies in all the $\Omega_n$. Let $g_n(z; p)$ be the Green's function for $\Omega_n$ with pole at $p$; this exists because of Theorem 5.2 and assumption (5.2c). If $m > n$, then $g_m - g_n$ is harmonic on $\Omega_n$ and is positive on $\partial\Omega_n$ and hence positive on all of $\Omega_n$. Hence, the sequence $\{g_n(z; p)\}$ is increasing on $\Omega_j \setminus \{p\}$ for each $j$ and so converges, uniformly on compact subsets of $\Omega \setminus \{p\}$ to either $+\infty$ identically or to a harmonic function $g(z; p)$. If the former occurs we say that $\Omega$ has no Green's function; more on this is given in Section 7 of this chapter. If the latter occurs, then it is a simple matter to check that $g(z; p)$ is independent of the particular choice of the regular exhaustion $\{\Omega_m\}$ and that $g(z; p)$ satisfies (5.1a) and (5.1b). Finally, if $\Omega$ already is known to have a Green's function satisfying (5.1a), (5.1b), (5.1c), then again it is immediate that the $g(z; p)$ arrived at by this process must coincide with it.

**Proposition 5.4.** Let $g$ be the Green's function for $\Omega$. Then for all points $p \neq q$ in $\Omega$ we have

$$g(p, q) = g(q, p) \tag{5.3}$$

***Proof.*** Assume first that $\Omega$ is a bounded domain each of whose boundary components is nontrivial. The function

$$g(z; q) + \log|z - q|$$

is harmonic on $\Omega$ and continuous at $\partial\Omega$ with boundary values equal to $\log|z - q|$, $z \in \partial\Omega$. Let $I(z; q) = \log|z - q|$ and let $\ell$ be the linear functional: $\ell(u) = \tilde{u}(p)$, where $\tilde{u}$ is the harmonic extension to $\Omega$ of $u$. Since $I$ is a harmonic function of $q$, so is $\ell(I(\cdot; q))$. But $\ell(I(\cdot; q)) = g(p; q) + \log|p - q|$. Consequently, $g(p; q)$ is a harmonic function of $q$ for $q \in \Omega$, $q \neq p$.

Next, set $d(p, q) = g(p; q) - g(q; p)$. $d$ is a harmonic function of both $p$ and $q$ except possibly where $p = q$. From the foregoing we see that $g(p; q)$ has a logarithmic pole at $p$ (as a function of $q$) and, of course, $g(q; p)$ also has a logarithmic pole at $p$ as a function of $q$. Hence, $d(p, z)$ is harmonic in a neighborhood of $z = p$. Further, if $x \in \partial\Omega$, we have

$$\liminf\{d(p, q) \colon q \to x\} = \liminf\{g(p, q) \colon q \to x\} \geqslant 0$$

and hence $d(p, q) \geqslant 0$ in $\Omega$ for each $p$ and all $q$. On the other hand,

$$\limsup\{d(p,q): p \to x\} = \limsup\{-g(q,p): p \to x\} \leqslant 0$$

so that $d(p, q) \leqslant 0$ in $\Omega$ for each $q$ and all $p$. Thus, $d \equiv 0$ and we see that $g(p, q) = g(q, p)$.

For the general case let $\{\Omega_m\}$ be a regular exhaustion of $\Omega$; there's no loss in assuming that $p, q \in \Omega_1$. If $g_m$ is the Green's function for $\Omega_m$ then we know $\{g_m(z; w)\}$ increases to $g(z; w)$, $z, w \in \Omega$ and $z \neq w$. Thus,

$$g(p,q) = \lim g_m(p,q) = \lim g_m(q,p) = g(q,p)$$

## 1.6. HARMONIC MEASURE

Let $\Omega$ be a domain on the sphere for which the Dirichlet problem is solvable and let $p$ be a point of $\Omega$. For each real-valued continuous function $u$ on $\partial\Omega = \Gamma$ we can associate the real number $\tilde{u}(p)$, where $\tilde{u}$ is the harmonic extension to $\Omega$ of $u$. The rule $u \mapsto \tilde{u}(p)$ is linear and, because of the maximum principle, we know $|\tilde{u}(p)| \leqslant \|u\|_\Gamma = \sup\{|u(z)|: z \in \Gamma\}$. Thus, the Riesz representation theorem implies that there is a unique real measure $\omega_p$ on $\Gamma$ with

$$\tilde{u}(p) = \int_\Gamma u\, d\omega_p, \qquad u \in C_r(\Gamma) \tag{6.1}$$

This measure is the *harmonic measure* on $\Gamma$ for $p$. Note as well that if $u \geqslant 0$ then $\tilde{u}(p) \geqslant 0$ and so $\omega_p$ is a non-negative measure; further, the total mass of $\omega_p$ is 1 since

$$\|\omega_p\| = \int_\Gamma 1\, d\omega_p = \tilde{1}(p) = 1$$

Clearly $\omega_p$ depends on $\Omega$ but we shall suppress this in the notation except when needed.

The next theorem relates the measures $\omega_p$ and $\omega_q$ for $p, q \in \Omega$.

**Theorem 6.1.** *If $p, q \in \Omega$, then $\omega_p$ and $\omega_q$ are boundedly mutually absolutely continuous. Indeed, if $\mathbf{K}$ is a compact set in $\Omega$, then there is a constant $M$ with*

$$\omega_q(E) \leqslant M\omega_p(E)$$

*for all $q \in \mathbf{K}$ and for all measurable sets $E$ in $\Gamma$.*

***Proof.*** We first show that $\omega_p$ and $\omega_q$ are mutually absolutely continuous. Suppose $E$ is a closed set in $\Gamma$ of $\omega_p$-measure 0. Let $u$ be a continuous function on $\Gamma$ with $u = 1$ on $E$ and $0 \leqslant u < 1$ off $E$. Let $v_n$ be the harmonic extension to $\Omega$ of $u^n$. Then $\{v_n\}$ is a decreasing sequence of positive harmonic functions on $\Omega$ and

$$v_n(p) = \int_\Gamma u^n \, d\omega_p \to \omega_p(E) = 0$$

Hence, $v_n \to 0$ on all of $\Omega$ so that

$$0 = \lim v_n(q) = \lim \int_\Gamma u^n \, d\omega_q = \omega_q(E)$$

For $p \in \Omega$ fixed, let us write $d\omega_q = A_q \, d\omega_p$, where $A_q$ is a non-negative function in $L^1(\omega_p)$. We wish to show that $A_q$ is actually in $L^\infty(\omega_p)$ and, indeed, its sup norm is uniformly bounded as $q$ varies over $K$.

The key ingredient in the proof is *Harnack's inequality*: there are positive numbers $c_1$ and $c_2$ depending only on **K** and $p$ with the property that if $v$ is a positive harmonic function on $\Omega$ satisfying

$$v(p) = 1, \quad \text{then} \quad c_1 \leqslant v(q) \leqslant c_2, \qquad q \in \mathbf{K}$$

It suffices to prove this when **K** is a closed disc containing $p$. Let $\mathfrak{D}$ be an open disc containing both $p$ and **K**. Suppose the assertion about the upper bound $c_2$ is false. Then there is a sequence $\{v_n\}$ of positive harmonic functions on $\Omega$ and a sequence $\{q_n\}$ of points in **K** with $v_n(p) = 1$ but $v_n(q_n) > n$. There is no loss in assuming that $q_n \to q \in \mathbf{K}$. Consider the holomorphic functions

$$f_n = \exp[-v_n - i\,{}^*v_n]$$

where ${}^*v_n$ is the harmonic conjugate of $v_n$ in $\mathfrak{D}$ vanishing at $p$. These functions $f_n$ are bounded by 1 and so at least a subsequence converges uniformly on compact sets in $\mathfrak{D}$ to a holomorphic function $f$. Since $f_n(p) = e^{-1}$ for all $n$, we have $f(p) = e^{-1}$ and so $f$ is not zero anywhere on $\mathfrak{D}$. But $|f_n(q_n)| \leqslant e^{-n}$ so that $f(q) = 0$, a contradiction. In a similar fashion the lower bound $c_1$ can be shown to exist.

With this observation, the proof of the theorem is quite easy. Let $E$ be a closed set in $\Gamma$ and suppose $\omega_p(E) = \sigma > 0$. Again let $u$ be a continuous function on $\Gamma$ with $u = 1$ on $E$ and $0 \leqslant u < 1$ off $E$. As before, let $v_n$ be the harmonic extension to $\Omega$ of $u^n$ and let $v$ be the limit of the sequence $\{v_n\}$. Then $v$ is a positive harmonic function in $\Omega$ and

$$v(p) = \lim \int_\Gamma u^n \, d\omega_p = \int_E d\omega_p = \omega_p(E) = \sigma$$

Thus,

$$\sigma c_1 \leqslant v(q) \leqslant \sigma c_2 \quad \text{for all } q \in \mathbf{K}.$$

However,

$$v(q) = \lim v_n(q) = \lim \int u^n \, d\omega_q$$

$$= \int_E d\omega_q = \omega_q(E)$$

Consequently,

$$\sigma c_1 \leqslant \omega_q(E) \leqslant \sigma c_2, \qquad q \in \mathbf{K}$$

Equivalently,

$$c_1 \leqslant \frac{\omega_q(E)}{\omega_p(E)} \leqslant c_2 \tag{6.2}$$

Since (6.2) holds for all closed sets $E$ in $\Gamma$ and since the measures are regular, (6.2) also holds for all measurable sets and this is exactly the assertion of the theorem.

**Proposition 6.2.** Suppose $\Omega_1$ and $\Omega_2$ are two domains and $h$ is a holomorphic function on $\Omega_1$ which maps $\text{CL}(\Omega_1)$ homeomorphically onto $\text{CL}(\Omega_2)$. Suppose the Dirichlet problem is solvable in $\Omega_1$ (and hence in $\Omega_2$); let $p_1 \in \Omega_1$ and put $p_2 = h(p_1) \in \Omega_2$. Let $\omega_1$ be harmonic measure on $\partial\Omega_1$ for $p_1$, and define a measure $\mu$ on $\partial\Omega_2$ by the rule

$$\mu(E) = \omega_1\big(h^{-1}(E)\big), \qquad E \subset \partial\Omega_2 \tag{6.3}$$

Then $\mu$ is harmonic measure on $\partial\Omega_2$ for the point $p_2$.

***Proof.*** The proof is shorter than the statement. The rule (6.3) is equivalent to the statement

$$\int_{\partial\Omega_2} u \, d\mu = \int_{\partial\Omega_1} u \circ h \, d\omega_1, \qquad u \in C(\partial\Omega_2) \tag{6.4}$$

Hence,

$$\int_{\partial\Omega_2} u \, d\mu = \widetilde{u \circ h}(p_1) = \tilde{u}\big(h(p_1)\big) = \tilde{u}(p_2)$$

which is just what was to be proved.

**Theorem 6.3.** *Harmonic measure has no atoms.*

***Proof.*** Fix some $p \in \Omega$ and let $\omega$ be harmonic measure for $p$. There is no loss in assuming that $\infty \in \Omega$ and that $0 \in \Gamma = \partial\Omega$. We shall show that $\omega$ has no mass at 0; indeed, that $\log|z|$ is in $L^1(\Gamma, \omega)$.

Let $g(z; \infty)$ be the Green's function with pole at $\infty$ and put $u(z) = g(z; \infty) - \log|z|$. Then $u$ is harmonic in $\Omega$ and $u$ is continuous at all points of $\Gamma \setminus \{0\}$ with value equal to $-\log|x|$. Let $\{f_n\}$ be a sequence of continuous functions on $\Gamma$ with $f_1 \leqslant f_2 \leqslant \cdots$, $\lim f_n(x) = -\log|x|$ if $x \in \Gamma$, $x \neq 0$, and $\{f_n(0)\}$ increasing to $\infty$. Let $u_n$ be the harmonic extension of $f_n$ to $\Omega$. Then $u_1 \leqslant u_2 \leqslant \cdots$ in $\Omega$ and $u_n(z) \leqslant u(z)$ if $z \in \Omega$ since $u_n \leqslant u$ on $\Gamma \setminus \{0\}$ and surely $u_n(z) < u(z)$ if $|z|$ is small enough. Thus,

$$u(p) \geqslant \lim u_n(p) = \lim \int_\Gamma f_n \, d\omega$$

$$= \lim[f_n(0)\,\omega(\{0\})] + \int_\Gamma -\log|x| \, d\omega(x)$$

which proves that $\omega(\{0\}) = 0$ and $\log|x|$ is in $L^1(\Gamma, \omega)$.

When the domain $\Omega$ has a nice boundary, harmonic measure is particularly easy to understand. We shall have great use for the next few results in Chapters 4 and 5.

**Theorem 6.4.** *Suppose $\Omega$ is bounded by a finite number of disjoint analytic simple closed curves. For each $p \in \Omega$ we have*

$$d\omega_p = \frac{-1}{2\pi}\frac{\partial}{\partial n} g(\cdot\,; p)\, ds \tag{6.5}$$

*where $g(\cdot\,; p)$ is the Green's function for $\Omega$ with pole at $p$, $\partial/\partial n$ is the derivative in the direction of the outward normal at $\Gamma$, and $ds$ is arc length.*

***Proof.*** The proof is an application of Green's theorem. Let $h$ be a smooth function on $\Gamma$ and for small $\delta > 0$ let

$$\Omega_\delta = \Omega \setminus \{z : |z - p| \leqslant \delta\}$$

Let $u$ be the harmonic extension to $\Omega$ of $h$ and let $v(z) = g(z; p)$. Then by Green's theorem on $\Omega_\delta$,

$$\iint_{\Omega_\delta} [u\Delta v - v\Delta u]\, dx\, dy = \int_{\partial\Omega_\delta} \left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right] ds \tag{6.6}$$

Now the left-hand side of (6.6) is zero since both $u$ and $v$ are harmonic on $\Omega_\delta$.

Further, $v = 0$ on $\Gamma$, so (6.6) simplifies to

$$\int_\Gamma h\frac{\partial v}{\partial n}\,ds = \int_{|z-p|=\delta}\left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right] ds$$

Let us set $z = p + re^{it}$, $0 \leqslant r \leqslant \delta$, and $0 \leqslant t \leqslant 2\pi$. Then the normal derivative is just the radial derivative and $g(z;\,p) = -\log r + G(z)$ where $G$ is harmonic near $p$. Thus,

$$\frac{\partial g}{\partial n} = -\frac{1}{r} + \text{continuous term}$$

Since $u$ is also continuous near $p$, we have

$$\int_{|z-p|=\delta} u\frac{\partial g}{\partial n}\,ds = -\int_0^{2\pi} u(p + \delta e^{it})\,dt + 0(\delta)$$

$$\to -2\pi u(p) \quad \text{as } \delta \to 0$$

Finally, the other term

$$-\int_{|z-p|=\delta} g\frac{\partial u}{\partial n}\,ds$$

behaves like $-\delta\log\delta$ as $\delta \to 0$ and so goes to zero as $\delta \to 0$.

It is convenient to put here two more facts related to harmonic measure. Let us assume that $\Gamma = \partial\Omega$ consists of $m + 1$ disjoint analytic simple closed curves. Let $p \in \Omega$ and let $h(z) = h(z;\,p)$ be the harmonic conjugate of $g(z;\,p)$, the Green's function for $\Omega$ and $p$. Of course, $h$ is not single-valued since it has periods about the holes in $\Omega$ (we will explore this type of difficulty later in some detail) but locally $g + ih$ is analytic and its derivative is single-valued on $\Omega$. We write $Q = g + ih$ and $Q'$ for its complex derivative.

**Proposition 6.5.** $d\omega_p(\zeta) = \dfrac{i}{2\pi}Q'(\zeta)\,d\zeta$

***Proof.*** Let $n$ be a unit outward normal at $\zeta \in \Gamma$, and let $\tau$ be a unit tangent at $\zeta$. By the Cauchy–Riemann equations

$$\frac{\partial h}{\partial n}(\zeta) = \frac{\partial g}{\delta\tau}(\zeta) = 0$$

since $g$ vanishes on $\Gamma$. Next, note that $n = -i\tau = -i\,d\zeta/|d\zeta|$ and

$$Q'(\zeta) = \lim_{t\downarrow 0}\frac{Q(\zeta + tn) - Q(\zeta)}{tn}$$

Further,

$$\frac{\partial g}{\partial n} = \lim_{t \downarrow 0} \frac{g(\zeta + tn) - g(\zeta)}{t}$$

Thus,

$$\begin{aligned} iQ'(\zeta)\, d\zeta &= i \lim_{t \downarrow 0} \frac{Q(\zeta + tn) - Q(\zeta)}{tn} d\zeta \\ &= -\lim_{t \downarrow 0} \frac{g(\zeta + tn) - g(\zeta)}{t} |d\zeta| \\ &= -\frac{\partial g}{\partial n} ds \end{aligned}$$

which establishes the result in light of Theorem 6.4.

**Proposition 6.6.** Let $\Gamma = \partial\Omega$ consist of $m + 1$ disjoint analytic simple closed curves, let $p \in \Omega$, and let $Q(z) = g(z; p) + ih(z; p)$ where $h$ is the (multiple-valued) harmonic conjugate of $g$. Then

(a) $Q'$ does not vanish on $\Gamma$
(b) $Q'$ has precisely $m$ zeros in $\Omega$, counting multiplicities

***Proof.*** We know that $Q'$ has a single pole of order one at $p$. Further, from Proposition 6.5 we also know that $iQ'(z)\,dz$ is a non-negative measure on $\Gamma$. Thus, the total change in $1/2\pi \arg Q'(z)$ as $z$ traverses $\Gamma$ once is precisely $m - 1$. But this must be exactly the number of zeros of $Q'$ minus the number of poles, and hence (b) is proven. To see (a) note that the Cauchy–Riemann equations imply that $h$ is increasing locally on $\Gamma$ and thus $Q$ must be one-to-one in a neighborhood of each $\zeta \in \Gamma$.

**Definition.** The $m$ zeros of $Q'$ in $\Omega$ are the *critical points* of the Green's function $g(\cdot; p)$. They obviously depend on $p$ but we suppress this in the notation.

The critical points of the Green's function will reappear in several different contexts in the succeeding material.

## 1.7. LOGARITHMIC CAPACITY

This section contains a quick trip through some of the most basic theorems in potential theory with the view of gaining some knowledge which will be helpful later on, as well as to become more familiar with the functions from the first part of the chapter.

Let $\mathbf{E}$ be a compact set in the plane and let $\mathcal{P}(\mathbf{E})$ denote the set of probability measures on $\mathbf{E}$: $\nu \in \mathcal{P}(\mathbf{E})$ if $\nu$ is a non-negative measure of total mass 1. For $\nu \in \mathcal{P}(\mathbf{E})$, set

$$p_\nu(z) = -\int_{\mathbf{E}} \log|z-\zeta|\,d\nu(\zeta), \qquad z \in \mathbf{C} \tag{7.1}$$

$p_\nu$ is the *potential generated* by $\nu$. Some properties of the potential follow.

$$p_\nu \text{ is harmonic off } \mathbf{E} \text{ except at } \infty \text{ where } p_\nu(z) + \log|z| \text{ is harmonic} \tag{7.2}$$

$$-p_\nu \text{ is subharmonic on } \mathbf{C} \tag{7.3}$$

$$p_\nu(z) \leqslant \sup\{\,p_\nu(\zeta) : \zeta \in E\} \quad \text{if } z \notin \mathbf{E} \tag{7.4}$$

It is easy to verify (7.2). As for (7.3), let

$$L_n(w) = \max\{\log|w|, -n\}$$

Then $L_n$ is subharmonic and continuous on $\mathbf{C}$ and $L_1 \geqslant L_2 \geqslant \cdots$. Set

$$q_n(z) = \int L_n(z-\zeta)\,d\nu(\zeta)$$

Then $q_n$ is continuous, subharmonic, and $\{q_n(z)\}$ decreases to $-p_\nu(z)$ for each $z$. Thus, (7.3) holds by Proposition 3.2.

To see that (7.4) holds, there is nothing lost in assuming that $\sup\{\,p_\nu(\zeta) : \zeta \in \mathbf{E}\}$ is finite. Let $M$ denote this supremum and note that since $M$ is finite, $\nu$ can have no point masses. Given $\varepsilon > 0$ and $\zeta_0 \in \mathbf{E}$ choose $\delta > 0$ so that the mass of $\nu$ on the disc $\mathbf{V} = \{\zeta : |\zeta - \zeta_0| \leqslant \delta\}$ is no more than $\varepsilon$. Put

$$p_1(z) = -\int_{\mathbf{V}} \log|z-\zeta|\,d\nu(\zeta)$$

$$p_2(z) = -\int_{\mathbf{E}\setminus\mathbf{V}} \log|z-\zeta|\,d\nu(\zeta).$$

$p_2$ is continuous on $\mathbf{V}$. Let $z \notin \mathbf{E}$ and let $\zeta'$ be a point of $\mathbf{E}$ at minimum distance to $z$. Then $\zeta' \to \zeta_0$ as $z \to \zeta_0$ and further, if $\zeta \in \mathbf{E}$, we have

$$|\zeta - \zeta'| \leqslant |\zeta' - z| + |z - \zeta| \leqslant 2|z-\zeta|$$

This gives

$$p_1(z) \leqslant \nu(\mathrm{V})\log 2 + p_1(\zeta')$$
$$< \varepsilon \log 2 + p_1(\zeta')$$

Thus, for $z$ near $\zeta_0$, we find

$$p_\nu(z) = p_1(z) + p_2(z) < \varepsilon \log 2 + p_1(\zeta') + p_2(\zeta') + \varepsilon$$
$$= \varepsilon(1 + \log 2) + p_\nu(\zeta')$$
$$\leqslant \varepsilon(1 + \log 2) + M$$

This proves that $\limsup\{p_\nu(z): z \to \zeta_0\} \leqslant M$ for all $\zeta_0 \in \mathrm{E}$ and hence (7.4) holds.

**Proposition 7.1.** Let $\Omega$ be a domain containing $\infty$ on which the Dirichlet problem is solvable and let $\omega_\infty$ be harmonic measure on $\partial\Omega$ for $\infty$. Let $g(z, \infty)$ be the Green's function for $\Omega$ with pole at $\infty$ and put

$$\gamma = \lim\{g(z, \infty) - \log|z| : |z| \to \infty\} \tag{7.5}$$

Then the potential generated by $\omega_\infty$ satisfies

$$-\int \log|z - \zeta|\, d\omega_\infty(\zeta) = \begin{cases} \gamma - g(z; \infty), & z \in \Omega \\ \gamma, & z \notin \Omega \cup \partial\Omega \end{cases} \tag{7.6}$$

and

$$-\int \log|z - \zeta|\, d\omega_\infty(\zeta) \leqslant \gamma \quad \text{for all } z \tag{7.7}$$

***Proof.*** Fix any point $a \in \Omega$, $a \neq \infty$. Set

$$w(z) = \int \log|a - \zeta|\, d\omega_z(\zeta) - \log|a - z| + g(z; \infty)$$

The function $w$ has a logarithmic pole at $a$, is harmonic on $\Omega \setminus \{a\}$, including at $\infty$, and $w$ vanishes identically on $\partial\Omega$. Thus, $w(z) = g(z; a)$. However, $g(z; a) = g(a; z)$ by Proposition 5.4 so we have

$$-\int \log|a - \zeta|\, d\omega_z(\zeta) = \{g(z; \infty) - \log|z - a|\} - g(a; z) \tag{7.8}$$

In (7.8) let $|z| \to \infty$; the term in the curly brackets approaches $\gamma$ so that

$$-\int \log|a - \zeta|\, d\omega_\infty(\zeta) = \gamma - g(a; \infty)$$

which is the desired equality in (7.6) for $a \in \Omega$. Next suppose $a \notin \Omega \cup \partial\Omega$; put

$$w(z) = -\int \log|a - \zeta|\, d\omega_z(\zeta) + \log|z - a| - g(z; \infty)$$

Then $w$ is harmonic in $\Omega$ and identically zero on $\partial\Omega$. Thus $w$ vanishes identically and, in particular

$$0 = w(\infty) = -\int \log|a - \zeta|\, d\omega_\infty(\zeta) - \gamma$$

which is the desired equality in (7.6) for $a \notin \Omega \cup \partial\Omega$. Lastly, (7.7) follows from (7.6) and the fact that a potential is lower semicontinuous.

**Definition.** The number $\gamma$ defined by (7.5) is the *Robin's constant* of $\Omega$, or the *Robin's constant* of $\mathbf{E} = \Omega^c$.

Now let $\mathbf{E}$ be any compact set in $\mathbf{C}$ and let $\Omega$ be the unbounded component of the complement of $\mathbf{E}$. Let $\{\Omega_n\}$ be any regular exhaustion of $\Omega$ and let $\{g_n(z; p)\}$ be the corresponding Green's function with pole at $p$. We noted in Section 5 that the sequence $\{g_n(z; p)\}$ is increasing with $n$ for each $z$ and converges uniformly on compact subsets of $\Omega$ to either $+\infty$ identically or to a function $g(z; p)$ which possesses properties (5.1a) and (5.1b). If the latter occurs for some $p \in \Omega$ then it occurs for all $p \in \Omega$ and the function $g(z; p)$ is independent of the sequence $\{\Omega_n\}$. In this case we know that

$$\gamma_n = \lim\{g_n(z; \infty) - \log|z| : |z| \to \infty\}$$

increases to

$$\gamma = \lim\{g(z; \infty) - \log|z| : |z| \to \infty\} \tag{7.9}$$

and this number is surely finite. Furthermore, if $\omega_\infty^{(n)}$ is harmonic measure on $\partial\Omega_n$ for $\infty$, then at least some subsequence of $\{\omega_\infty^{(n)}\}$ converges weak-* in the space of measures to a probability measure $\nu$ supported on $\mathbf{E}$ (indeed, on $\partial\Omega$) which satisfies

$$p_\nu(z) = \begin{cases} \gamma - g(z; \infty), & z \in \Omega \\ \gamma, & z \notin \Omega \cup \partial\Omega \end{cases} \tag{7.10}$$

In this case we define the *logarithmic capacity of* $\mathbf{E}$ by

$$\operatorname{cap}(\mathbf{E}) = e^{-\gamma} \tag{7.11}$$

What, however, happens in the former case, when $g_n(z; p)$ increases to $+\infty$ for each $z, p \in \Omega$? In this case we must have $\{\gamma_n\}$ increasing to $+\infty$ [or else $g_n(z, \infty)$ would stay bounded] and we set

$$\operatorname{cap}(\mathbf{E}) = 0$$

Note well that the capacity of $\mathbf{E}$ depends only on the unbounded component of the complement of $\mathbf{E}$. We can phrase this in the following way:

$$\operatorname{cap}(\mathbf{E}) = \operatorname{cap}(\hat{\mathbf{E}}) \tag{7.12}$$

where $\hat{\mathbf{E}}$ is the *polynomial hull* of $\mathbf{E}$ defined by the rule

$$z \in \hat{\mathbf{E}} \text{ if and only if } |p(z)| \leqslant \max\{|p(\zeta)| : \zeta \in \mathbf{E}\}$$

for all polynomials $p$. We now show that $\hat{\mathbf{E}}$ is obtained from $\mathbf{E}$ by the device of adding to $\mathbf{E}$ all the bounded components of the complement of $\mathbf{E}$. First, if $\mathcal{O}$ is a bounded component of $\mathbf{E}^c$, then the maximum principle immediately implies that each point of $\mathcal{O}$ lies in $\hat{\mathbf{E}}$. Second, if $|z_0| > \max\{|\zeta| : \zeta \in \mathbf{E}\}$, then $z_0 \notin \hat{\mathbf{E}}$ since the polynomial $p(z) = z$ is larger at $z_0$ than it is on $\mathbf{E}$. Let $\Omega$ be the unbounded component of the complement of $\mathbf{E}$. Clearly the set of those points of $\Omega$ which are not in $\hat{\mathbf{E}}$ is open. We shall also show that $\Omega \cap \hat{\mathbf{E}}$ is open and thus $\Omega \cap \hat{\mathbf{E}}$ is both open and closed and consequently empty. Let $a \in \Omega \cap \hat{\mathbf{E}}$; then the functional $p \mapsto p(a)$ defined on all polynomials is bounded, with norm 1, since $|p(a)| \leqslant \|p\|_{\mathbf{E}}$. Hence, it can be extended to a linear functional on $C(\mathbf{E})$ of norm 1; this implies that there is a measure $\lambda$ on $\mathbf{E}$ of total variation 1 with

$$\int_{\mathbf{E}} p \, d\lambda = p(a), \qquad p \text{ a polynomial}$$

The function

$$u(b) = \int_{\mathbf{E}} \frac{z - a}{z - b} \, d\lambda(z)$$

is continuous for $b \in \Omega$ and equals 1 at $a$; let $b$ be any point near $a$ at which $u(b) \neq 0$. If $p$ is a polynomial then so is

$$\frac{p(z) - p(b)}{z - b}(z - a)$$

and we find

$$0 = \int_{\mathbf{E}} \frac{p(z) - p(b)}{z - b}(z - a)\, d\lambda(z)$$

or

$$p(b) = c\int_{\mathbf{E}} p(z)\frac{z - a}{z - b}\, d\lambda(z), \qquad c = (u(b))^{-1}$$

Consequently,

$$|p(b)| \leqslant C\|p\|_{\mathbf{E}}$$

where $C$ is a constant depending only on $a$ and $b$, not $p$. Replace $p$ by $p^n$, take $n$th roots, and then let $n \to \infty$. We find

$$|p(b)| \leqslant \|p\|_{\mathbf{E}}$$

That is, $b \in \Omega \cap \hat{\mathbf{E}}$ if $b$ is near $a$. This shows $\Omega \cap \hat{\mathbf{E}}$ is open and, as above, $\Omega \cap \hat{\mathbf{E}}$ must be empty. In summary, we have this result.

**Proposition 7.2.** Let $\mathbf{E}$ be compact, and set

$$\hat{\mathbf{E}} = \{z : |p(z)| \leqslant \|p\|_{\mathbf{E}} \text{ for all polynomials } p\}$$

Then $\hat{\mathbf{E}}$ is obtained by adding to $\mathbf{E}$ all the bounded components of $\mathbf{E}^c$. Furthermore, cap($\mathbf{E}$) = cap($\hat{\mathbf{E}}$).

We now derive another approach to the logarithmic capacity of a compact set.

Let $\mathbf{E}$ be a compact set and for each positive integer $n$ set

$$M_n = \inf\left\{ \sup_{z\in\mathbf{E}} \prod_1^n |z - z_j| : z_1, \ldots, z_n \in \mathbf{C}\right\} \tag{7.13}$$

That is, $M_n$ is the smallest that the sup norm on $\mathbf{E}$ of a monic polynomial of degree $n$ can be made. We leave it as an exercise to show that the infimum in (7.13) is actually a minimum and that if $z_1^*, \ldots, z_n^*$ are points for which

$$M_n = \sup_{z\in\mathbf{E}} \prod_1^n |z - z_j^*|$$

then $z_1^*, \ldots, z_n^*$ lie in the closed convex hull of $\mathbf{E}$. Next we observe that

$$M_{n+m} \leqslant M_n M_m \quad \text{for all } n, m \tag{7.14}$$

since the union best points for $n$ and the best points for $m$ form one possible set of points for $n + m$. We now need a simple fact:

suppose $c_n + c_m \geqslant c_{n+m}$ for all $n, m$; then $\lim c_m/m$ exists although it may be $-\infty$ (7.15)

To see (7.15), let $\alpha > \inf\{c_m/m\}$. Then choose an integer $s$ with $\alpha > c_s/s$. For $m > s$ we write $m = us + v$, where $0 \leqslant v < s$. Hence

$$c_m = c_{us+v} \leqslant c_{us} + c_v \leqslant uc_s + c_v$$

Thus,

$$\frac{c_m}{m} \leqslant \frac{u}{us + v} c_s + \frac{1}{m} c_v$$

As $m \to \infty$, we know $u \to \infty$; we then have for large $m$

$$\frac{c_m}{m} \leqslant \frac{c_s}{s} + \varepsilon < \alpha + \varepsilon$$

Consequently,

$$\inf_m \frac{c_m}{m} \leqslant \limsup\left\{\frac{c_m}{m} : m \to \infty\right\}$$

$$\leqslant \frac{c_s}{s} \leqslant \alpha$$

Since this holds for all $\alpha > \inf\{c_m/m\}$ we find that

$$\lim\left\{\frac{c_m}{m} : m \to \infty\right\} = \inf\left\{\frac{c_m}{m}\right\}$$

With this in mind and in view of (7.14) we see that

$$\rho(\mathbf{E}) = \lim\{M_m^{1/m} : m \to \infty\} \tag{7.16}$$

exists, although it may be 0.

The function $\rho$ is a set function, defined at least on compact sets and $\rho$ has these two properties

$$\text{if } \mathbf{E} \subset \mathbf{F}, \quad \text{then } \rho(\mathbf{E}) \leqslant \rho(\mathbf{F}) \tag{7.17}$$

if $\{\mathbf{E}_n\}$ is a decreasing sequence of compact sets with $\mathbf{E} = \cap \mathbf{E}_n$, then $\rho(\mathbf{E}) = \lim \rho(\mathbf{E}_n)$ (7.18)

The proof of (7.17) is trivial so we examine (7.18). Given $\varepsilon > 0$ there is a large integer $s$ such that

$$\varepsilon + \rho(\mathbf{E}) \geqslant M_s^{1/s}(\mathbf{E})$$

However, with this $s$ fixed, we know that

$$M_s^{1/s}(\mathbf{E}) \geqslant M_s^{1/s}(\mathbf{E}_n) - \varepsilon$$

for all large enough $n$ and also that

$$M_s^{1/s}(\mathbf{E}_n) \geqslant \rho(\mathbf{E}_n), \qquad \text{all } s$$

since $\rho(\mathbf{G}) = \inf_r\{M_r^{1/r}(\mathbf{G})\}$ for any compact set $\mathbf{G}$. Putting these inequalities together we find

$$2\varepsilon + \rho(\mathbf{E}) \geqslant \rho(\mathbf{E}_n), \qquad n \text{ large}$$

so that (7.18) holds.

**Theorem 7.3.** $\rho = e^{-\gamma} = \operatorname{cap}(\mathbf{E})$

***Proof.*** Let $z_1^*, \ldots, z_n^*$ be an optimal choice in (7.13). Then

$$\log M_n \geqslant \sum_1^n \log|z - z_j^*| \qquad \text{if} \qquad z \in \mathbf{E}$$

and so if we define

$$v(z) = -\frac{1}{n}\sum_1^n \log|z - z_j^*|$$

we find

$$v(z) \geqslant -\frac{1}{n}\log M_n, \qquad z \in \mathbf{E}$$

Put $\mathbf{F} = \{z \in \mathbf{C} : v(z) \geqslant -(1/n)\log M_n\}$. Then $\mathbf{F}$ is compact and $\mathbf{E} \subset \mathbf{F}$. Consequently,

$$\gamma(\mathbf{E}) \geqslant \gamma(\mathbf{F}) \tag{7.19}$$

However, we notice that $-v(z) - (1/n)\log M_n$ has all the properties of the Green's function of the complement of $\mathbf{F}$ with pole at $\infty$ so that

$$-v(z) - \frac{1}{n}\log M_n = g(z, \infty; \mathbf{F}^c)$$

Thus,

$$\begin{aligned}\gamma(\mathbf{F}) &= \lim\{g(z, \infty; \mathbf{F}^c) - \log|z| : |z| \to \infty\} \\ &= -\frac{1}{n}\log M_n\end{aligned}$$

Consequently,

$$\gamma(\mathbf{E}) \geq \gamma(\mathbf{F}) = -\frac{1}{n} \log M_n$$

which implies

$$\operatorname{cap}(\mathbf{E}) = e^{-\gamma(\mathbf{E})} \leq \rho(\mathbf{E})$$

We prove the reverse inequality first for a compact set **E** with the property that $\Omega$, the unbounded component of the complement of **E**, is regular for the Dirichlet problem; that is, the Dirichlet problem is solvable in $\Omega$. Suppose, then that $\Omega$ is such a domain, $\infty \in \Omega$, and **E** is the complement of $\Omega$. Let $g(z, \infty)$ be the Green's function for $\Omega$ with pole at $\infty$ and recall (7.6)

$$-\int \log|z - \zeta| \, d\omega_\infty(\zeta) = \gamma - g(z, \infty), \qquad z \in \Omega \tag{7.20}$$

Choose a compact set **K** in $\Omega$ so close to $\partial\Omega$ that

$$g(z, \infty) < \varepsilon, \qquad z \in \mathbf{K} \tag{7.21}$$

Since $\omega_\infty$ is a probability measure on $\partial\Omega \subset \mathbf{E}$ there is a sequence $\{\nu_n\}$ of measures of the form

$$\nu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{j,n}$$

where $\delta_{j,n}$ is the unit point mass at a point $\zeta_{j,n} \in \partial\Omega$, such that $\{\nu_n\}$ converges weak-* to $\omega_\infty$ in the space of measures on $\partial\Omega$; the points $\zeta_{jn}$ need not be distinct. Thus,

$$\int \log|z - \zeta| \, d\nu_n(\zeta) \to \int \log|z - \zeta| \, d\omega_\infty(\zeta)$$

uniformly on compact subsets of $\Omega$ and, in particular, on **K**. If **K** is chosen so that **E** lies in $\hat{\mathbf{K}}$ then

$$\max\{|q(z)| : z \in \mathbf{K}\} \geq \max\{|q(z)| : z \in \mathbf{E}\} \tag{7.22}$$

for any polynomial $q$. Set

$$Q_n(z) = \prod_{j=1}^{n} (z - \zeta_{jn})$$

Then

$$\left| \frac{1}{n} \log|Q_n(z)| - \int \log|z - \zeta|\, d\nu_n(\zeta) \right| < \varepsilon, \qquad z \in \mathbf{K}$$

for $n$ large enough and so, combining this inequality with (7.20), (7.21) and (7.22) we find

$$\rho(\mathbf{E}) \leqslant \max_{z \in \mathbf{E}} |Q_n(z)|^{1/n} \leqslant e^{-\gamma + 2\varepsilon}$$

Hence, Theorem 7.3 holds in the special case when the Dirichlet problem is solvable in the unbounded component of the complement of **E**. For the general case let $\{\Omega_n\}$ be a regular exhaustion of $\Omega$ and let $\mathbf{E}_n$ be the complement of $\Omega_n$. Then, by definition

$$e^{-\gamma_n} \to e^{-\gamma}$$

and $\rho(\mathbf{E}_n) \to \rho(\mathbf{E})$ by (7.18). This finishes the proof.

**Theorem 7.4.** *Let* **E** *be a compact set of capacity zero. Then there is an element* $\sigma \in \mathcal{P}(\mathbf{E})$ *such that*

$$\lim\{ p_\sigma(z) : z \to \mathbf{E}\} = \infty \tag{7.23}$$

***Proof.*** We begin by noting that the proof of Theorem 7.3 showed that when computing $M_n$ and $\rho$ attention could be restricted to monic polynomials all of whose zeros lie in **E** (indeed, in $\partial\Omega$, where $\Omega$ is the unbounded component of the complement of **E**.) Let $z_{1n}, \ldots, z_{nn}$ be points of **E** such that

$$\max_{z \in \mathbf{E}} \left( \prod_{j=1}^{n} |z - z_{jn}| \right)^{1/n} < \varepsilon_n$$

where $\varepsilon_n \to 0$ as $n \to \infty$, since cap(**E**) = 0. Thus,

$$-\frac{1}{n} \sum_{j=1}^{n} \log|z - z_{jn}| > -\log \varepsilon_n, \qquad z \in \mathbf{E}$$

Choose $n = n_k \to \infty$ such that $\varepsilon_{n_k} < \exp[-e^k]$ and put

$$\sigma_k = \frac{1}{n_k} \sum_{j=1}^{n_k} \delta_{j, n_k}, \qquad k = 1, 2, 3, \ldots$$

where $\delta_{j, n_k}$ is the unit point mass at $z_{j, n_k}$. Finally, put

$$\sigma = \sum_{1}^{\infty} \frac{1}{2^k} \sigma_k$$

so that $\sigma \in \mathcal{P}(\mathbf{E})$ and for $z \in \mathbf{E}$ we have

$$p_\sigma(z) = -\int \log|z - \zeta|\, d\sigma(\zeta) \geqslant \sum_1^\infty \frac{1}{2^k} e^k = \infty$$

By lower semicontinuity we find

$$\liminf\{ p_\sigma(z) : z \to \zeta, \zeta \in \mathbf{E}\} \geqslant p_\sigma(\zeta) = \infty$$

The final theorems of this section show that a compact set of capacity zero is "removable" for bounded harmonic functions and that this property characterizes such sets.

**Theorem 7.5.** *Let* $\mathbf{E}$ *be a compact set and* $\Omega$ *any domain containing* $\mathbf{E}$. *If* $\mathbf{E}$ *has logarithmic capacity zero then each function* $u$ *which is bounded and harmonic in* $\Omega \setminus \mathbf{E}$ *extends to be harmonic in* $\Omega$.

***Proof.*** Let $\Omega'$ be a domain for which the Dirichlet problem is solvable with $\mathbf{E} \subset \Omega'$ and $\mathrm{CL}(\Omega') \subset \Omega$. Let $h$ be the function harmonic on $\Omega'$, continuous on $\mathrm{CL}(\Omega')$, with $h = u$ on $\partial\Omega'$ and put $v = u - h$. Then $v$ is bounded and harmonic on $\Omega' \setminus \mathbf{E}$ and $v = 0$ on $\partial\Omega'$. Let $\sigma$ be the measure constructed in Theorem 7.4 and let $p(z)$ be the potential generated by $\sigma$. The function $v - \varepsilon p$ is harmonic on $\Omega' \setminus \mathbf{E}$ and is nonpositive on $\partial(\Omega' \setminus \mathbf{E}) \subseteq \partial\Omega' \cup \mathbf{E}$ (we may assume that the diameter of $\Omega$ is at most one with no loss of generality). Hence, $v \leqslant \varepsilon p$ on $\Omega' \setminus \mathbf{E}$ and, letting $\varepsilon \to 0$ we find that $v \leqslant 0$ on $\Omega' \setminus \mathbf{E}$. In a like fashion by considering $v + \varepsilon p$ we see that $v \geqslant 0$ on $\Omega' \setminus \mathbf{E}$. Hence, $v \equiv 0$ on $\Omega' \setminus \mathbf{E}$ and so $u = h$ on $\Omega' \setminus \mathbf{E}$. But $h$ is harmonic across $\mathbf{E}$ and thus $u$ is as well.

**Theorem 7.6.** *Let* $\mathbf{E}$ *be a compact set. Suppose* $\mathbf{E}$ *has the property that whenever* $\Omega$ *is a domain containing* $\mathbf{E}$ *and* $u$ *is bounded and harmonic on* $\Omega \setminus \mathbf{E}$, *then* $u$ *is actually harmonic on* $\Omega$. *Then* $\mathbf{E}$ *has logarithmic capacity zero.*

***Proof.*** The hypothesis certainly implies $\mathbf{E}$ contains no continuum (otherwise there is a nonconstant bounded analytic function in the complement of that continuum). Hence, $\Omega = S^2 \setminus \mathbf{E}$ is connected. Suppose cap(E) is positive; then $\Omega$ has a (generalized) Green's function $g(z, \infty)$ with pole at $\infty$. By hypothesis $g(z, \infty)$ extends to be harmonic across $\mathbf{E}$ and hence $g(z, \infty)$ is harmonic on $\mathbf{C}$ with a logarithmic pole at $\infty$. Let $h$ be the harmonic conjugate of $g(z, \infty)$ on $\mathbf{C}$ and consider

$$F = \exp[-g - ih]$$

$F$ is a bounded entire function and hence is constant; this contradicts the fact that $g(z, \infty)$ is not identically $\infty$. Thus, cap(E) $= 0$.

## ADDITIONAL READINGS AND NOTES

The material of Chapter 1 is classical and may be found in many texts. The paperback book of Fuchs (1967) has a nice development of the topics of this chapter and others as well; the presentation here is very much like his. The book of Tsuji (1959) is a compendium of results on function and potential theory in the plane. Helms (1969) gives an exposition of potential theory in Euclidean $n$-space. In the exercises which follow two more notions of capacity are developed and shown to coincide with the logarithmic capacity presented in Section 1.7; the concept of the transfinite diameter of a compact set was introduced and studied by Fekete. Among other things, he showed that the logarithmic capacity (equivalently, the transfinite diameter, $\tau$) is the only nonzero set function defined on the compact sets in the plane which satisfies these four properties:

1. $\tau(\mathbf{E}) \leqslant \tau(\mathbf{F})$ if $\mathbf{E} \subseteq \mathbf{F}$
2. If $a \in \mathbf{C}$ and $a\mathbf{E} = \{az : z \in \mathbf{E}\}$, then $\tau(a\mathbf{E}) = |a|\tau(\mathbf{E})$
3. If $\varepsilon > 0$ and $\mathbf{E}_\varepsilon = \{z : \operatorname{dist}(z, \mathbf{E}) \leqslant \varepsilon\}$, then $\tau(\mathbf{E}_\varepsilon) \to \tau(\mathbf{E})$ as $\varepsilon \to 0$
4. If $Q$ is a monic polynomial of degree $k$ and if $\mathbf{E}^*$ consists of all roots of $Q(z) = w$ as $w$ ranges over $\mathbf{E}$, then $\tau(\mathbf{E}^*) = (\tau(\mathbf{E}))^{1/k}$

See Fekete (1923).

## EXERCISES

**1.** Prove the assertion made in the introduction to this chapter, that the Dirichlet problem is not solvable on the punctured disc, $0 < |z| < 1$.

**2.** If $u$ is subharmonic on $a < |z| < b$, then

$$I(r) = \int_{-\pi}^{\pi} u(re^{it})\,dt, \qquad a < r < b$$

is a convex function of $\log r$.

**3.** Let $\Omega$ be a domain for which the Dirichlet problem is solvable and let $E$ be a $\omega$-measurable set in $\Gamma = \partial\Omega$. If $u$ is subharmonic on $\Omega$ and

$$\limsup\{u(z) : z \to \zeta\} \leqslant M, \qquad \text{all } \zeta \in E$$

$$\limsup\{u(z) : z \to \lambda\} \leqslant L, \qquad \text{all } \lambda \in \Gamma \setminus E$$

then

$$u(z) \leqslant M\omega_z(E) + L\omega_z(\Gamma \setminus E), \qquad z \in \Omega$$

4. If, in problem 3, $E = \Gamma$ and if $u(z_0) = M$ for some $z_0 \in \Omega$, show $u \equiv M$ in $\Omega$.
5. Prove a function $u$ on a compact set **K** with values in $[-\infty, \infty)$ is upper semicontinuous if and only if there are continuous functions $\{f_j\}$ on **K** with $f_1 \geqslant f_2 \geqslant \cdots$ and $f_j(x) \to u(x)$, $x \in \mathbf{K}$.
6. Suppose $\Omega$ is simply connected and $\Gamma = \partial\Omega$ has two or more points. Show that the Dirichlet problem is solvable in $\Omega$.
7. Let $\Omega$ be bounded and simply connected and let $g(z; z_0)$ be the Green's function for $\Omega$ with pole at $z_0$. If $H$ is the harmonic conjugate of $G(z) = g(z; z_0) + \log|z - z_0|$ on $\Omega$, show that $F(z) = (z - z_0)\exp[-G(z) - iH(z)]$ is a one-to-one mapping of $\Omega$ onto the open unit disc $\Delta$, with $F(z_0) = 0$.
8. Find the Green's function for the open unit disc $\Delta$ with pole at $a \in \Delta$. Do the same for the annulus $\{z : 1 < |z| < \rho\}$ with pole at $r \in (1, \rho)$.
9. Let **E** be compact. Show that there are points $z_1^*, \ldots, z_n^*$ such that

$$M_n = \sup\left\{\prod_1^n |z - z_j^*| : z \in \mathbf{E}\right\}$$

Show further that $z_1^*, \ldots, z_n^*$ lie in the convex hull of **E**.
10. Let **E** be a compact, connected set with connected complement $\Omega$ (relative to the sphere) and let $f(z) = Az + b_0 + b_1/z + \cdots$ be the Riemann mapping of $\Omega$ onto $\{w : |w| > 1\}$ with $f(\infty) = \infty$, $A > 0$. Show that $\operatorname{cap}(\mathbf{E}) = 1/|A|$. If $\mathbf{E} = [a, b]$ in $\mathbb{R}$, show $\operatorname{cap}(\mathbf{E}) = \frac{1}{4}(b - a)$.
11. Let $\Omega$ be a domain for which the Dirichlet problem is solvable and let $\mathbf{E} \subset \partial\Omega$ be compact. If $\operatorname{cap}(\mathbf{E}) = 0$, then the harmonic measure of **E** is also zero.
12. Let $\Omega_1$ and $\Omega_2$ be two domains on which the Dirichlet problem is solvable and suppose $\Omega_1 \subset \Omega_2$. Let $p \in \Omega_1$ and let $\omega_1$ and $\omega_2$ be harmonic measure on $\partial\Omega_1$ and $\partial\Omega_2$, respectively, for $p$. If $E$ is a compact set in $\partial\Omega_1 \cap \partial\Omega_2$, show that $\omega_1(E) \leqslant \omega_2(E)$.

There are at least two other ways to arrive at the logarithmic capacity of a compact set **E**. These are developed in the next several exercises.

Let $z_1, \ldots, z_n$ be points of **E** and set

a. $P(z_1, \ldots, z_n) = \prod_{j<k} |z_j - z_k|$

b. $P_n = \sup\{P(z_1, \ldots, z_n) : z_1, \ldots, z_n \in \mathbf{E}\}$

c. $d_n = P_n^{\varepsilon_n}$, $\varepsilon_n = 2/n(n-1)$

13. Show that $d_1 \geqslant d_2 \geqslant \cdots$. Let $\tau(\mathbf{E}) = \lim d_n$; $\tau$ is the *transfinite diameter* of **E**.