Solutions for test 1 (In class part)

① $z$, $z^3$, $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$

are analytic (represented as power series)

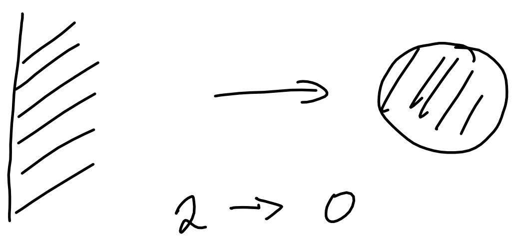
$|z|$, $\bar{z}^2$ are not

$$\frac{\partial \bar{z}^2}{\partial \bar{z}} = 2\bar{z} \not\equiv 0 \qquad \frac{\partial}{\partial \bar{z}} |z| = \frac{\partial}{\partial \bar{z}} (z\bar{z})^{1/2}$$

$$= \frac{1}{2} (z\bar{z})^{-1/2} \cdot z \not\equiv 0$$

(For $|z|$ see also #3)

②



Any such conf map has to be LFT

$2 \to 0$

$-2 \to \infty$

symmetric to $2$

so $\varphi(z) = c \, \frac{z-2}{z+2}$

Since $\left| \frac{ix-2}{ix+2} \right| = 1$ for $x \in \mathbb{R}$

then $|c| = 1$.

But only real $c$ preserve ~~say~~ real line

so $\varphi(z) = \pm \frac{z-2}{z+2}$ 2 possibilities

(3) No, any non-constant analytic function is an open map. So, if $w_0 \in \operatorname{Ran} f$ then for some $\varepsilon > 0$

$$\{w : |w - w_0| < \varepsilon\} \subset \operatorname{Ran} f$$

which is impossible if $\operatorname{Ran} f \subset \mathbb{R}$

(4). $\ln|z| = \frac{1}{2} \ln|z|^2 = \frac{1}{2} \ln(z\bar{z})$

$$\frac{\partial}{\partial z} \ln(z\bar{z}) = \frac{1}{z\bar{z}} \bar{z} = \frac{1}{z}$$

$$\frac{\partial}{\partial \bar{z}} \left(\frac{1}{z}\right) = 0, \quad \text{so} \quad \Delta \ln|z| = 0.$$

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Another way.

For any $z_0 \neq 0$, $\log z$ is defined in a neighborhood of $z_0$.

$$\operatorname{Re} \log z = \ln|z|$$

and real part of an analytic function is harmonic $\square$

(5) $u(z) = \dfrac{1}{|z^4-1|^{10}} = |f(z)|^2$

where $f(z) = \dfrac{1}{(z^4-1)^5}$ — analytic for $z^4 \neq 1$

$\cancel{\Delta |f(z)|^2 = |f'(z)|^2}$

$\Delta |f(z)|^2 = 4 \partial \bar{\partial} |f(z)|^2 = 4 |f'(z)|^2$

$= 4 \left| -5 (z^4-1)^{-6} \, 4z^3 \right|^2$

$= \cancel{80 |z|} \rightarrow 1600 \dfrac{|z|^6}{|z^4-1|^{12}}$

$1600$

(6) $f$ is defined everywhere except $\frac{\pi}{2} + \pi n$, $n \in \mathbb{Z}$. Therefore, the open disc centered at 0 of radius $\frac{\pi}{2}$ belongs to the domain, and the Taylor series about 0 converges in this disc

Therefore, $r \geq \pi/2$

On the other hand, if $r > \pi/2$, then $f$ has removable singularity at $z = \pi/2$. But we know that $f$ has a pole

at this point.
Therefore

$\therefore \quad r = \dfrac{\pi}{2}$

⑦ $$\int_{|z|=1} \frac{1}{|2z-1|^2}\,|dz|$$

Parametrization $\quad z = e^{it} \qquad dz = i e^{it}\,dt$

$$= iz\,dt$$

$$|dz| = dt = \frac{dz}{iz}$$

$$\frac{1}{|2z-1|^2} = \frac{1}{(2z-1)(2\bar{z}-1)}$$

$$= \frac{1}{(2z-1)(2/z - 1)} = \frac{z}{(2z-1)(2-z)}$$

$|z|=1$

So, we need to compute the integral

$$\frac{1}{i}\int_{|z|=1} \frac{dz}{(2z-1)(2-z)} = \frac{1}{2i}\int_{|z-1|} \frac{dz}{(2-z)(z-1/2)}$$

$z = \frac{1}{2}$ only singularity inside the contour

$$= \frac{1}{2i}\, 2\pi i \, \frac{1}{2 - 1/2} = \pi \frac{1}{3/2} = \boxed{\frac{2}{3}\pi}$$