MATH 2250 Class Notes

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Common representation

Complex numbers are commonly written,

$$z = x + iy,$$

where $i^2 = 1$ and $x, y \in \mathbb{R}$. Multiplication by i is commutative. Specifically,

$$iy = yi.$$

Multiplication follows the usual distributive rules:

$$(a + bi)(c + di) = (ac - bd) + i(bc + ad).$$

If we view a complex number z as just an ordered pair of real numbers (x, y), we can write this multiplication rule as

$$(a, b)(c, d) = (ac - bd, bc + ad).$$

Matrix representation

We can represent $z = x + iy$ as

$$x + iy \sim \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

Trivially, it follows that

$$i \sim \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$1 \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Check $i^2 = -1$:

$$i^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1$$

Structure of complex numbers

Definition 1 (Algebra)**.** *a vector space equipped with a multiplication operation*

The complex numbers are a commutative algebra.
To show complex numbers are a field, need to find inverse:

$$\begin{aligned}\frac{1}{x+iy} &= \frac{x-iy}{(x+iy)(x-iy)} \\ &= \frac{x-iy}{x^2+y^2} \\ &= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\end{aligned}$$

Notation

Given $z = x + iy$, we say $\bar{z} = x - iy$ is the complex conjugate of z and $|z| = \sqrt{x^2+y^2}$ is the modulus of z. We can easily see that

$$z^{-1} = \frac{\bar{z}}{|z|^2},$$

$$z^2 = z\bar{z},$$

and

$$zz^{-1} = \frac{z\bar{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1.$$

These notation and representations of complex numbers are almost always more convenient to work with than the ordered pair notation.

Polar form

If $z \neq 0$, then

$$z = |z|\frac{z}{|z|} = |z|(x+iy),$$

where $x^2 + y^2 = 1$. Thus, (x, y) must be on the unit circle and we can find an α such that

$$x = \cos\alpha, y = \sin\alpha.$$

α is unique up to $\alpha + 2\pi n$, where $n \in \mathbb{Z}$. The α such that $\alpha \in [0, 2\pi)$ is called the argument of z.

In the matrix notation, if $|z| = 1$, we have

$$z = \begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & -\cos\alpha\end{pmatrix}.$$

Multiplying by a unimodular complex number is equivalent to rotating by α.

Multiplication

When multiplying two complex numbers, we multiply their moduli and add their arguments.

We can write z by

$$z = |z|(\cos\alpha + i\sin\alpha) = |z|e^{i\alpha}.$$

This implies the identity

$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$

We have the identity,

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

There are several ways to prove this identity:

1. Trigonometric identites
2. Matrix notation
3. Identity $e^{i\alpha} = \cos\alpha + i\sin\alpha$ (must prove this)

Prove $e^{i\alpha} = \cos\alpha + i\sin\alpha$ using Taylor series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \text{ (definition for complex } z\text{)}$$

$$\sin z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}(-1)^n = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots$$

$$\cos z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}(-1)^n = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots$$