math.brown.edu/~treil,

9/3/10 Homework 1 (on ~~mycourses.brown.edu~~)

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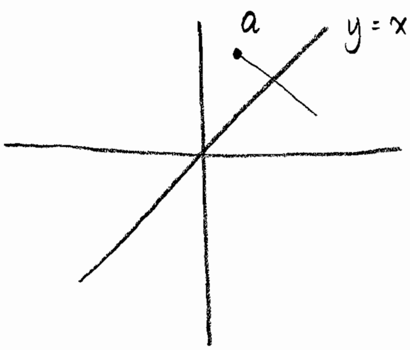

Reflect a

idea rotate it.

$$\overline{\left(ae^{-i\frac{\pi}{4}}\right)}\, e^{i\frac{\pi}{4}} = \bar{a}e^{i\frac{\pi}{4}} \cdot e^{i\frac{\pi}{4}}$$
$$= \bar{a}e^{i\frac{\pi}{2}}$$
$$= i\bar{a}$$

This problem was not in the HW

Def Analytic function (on an open set) is a function locally represent as converge power series.

Note "locally" means $\forall z_0 \in \Omega$ $\exists$ a neighborhood $\mathcal{U} \ni z_0$

"power series" $\rightsquigarrow f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$

"real analytic" $\rightsquigarrow \sum_{n,k=0}^{\infty} c_{n,k}(x-x_0)^n (y-y_0)^k$ $\rightsquigarrow$ also power series

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■ Review of Power Series

$$\sum_{n=0}^{\infty} c_n (z-z_0)^n \quad , \quad \sum_{n=0}^{\infty} c_n z^n$$

I. Radius of Convergence

Thm Given $\sum c_n z^n$ $\exists! R \in [0,\infty]$ such that

1.) $\forall z : |z| < R$, $\sum c_n z^n$ converges

2). $\forall z : |z| > R$, $\sum_0^\infty c_n z^n$ diverges.

R - radius of convergence

$$\frac{1}{R} = \limsup |a_n|^{\frac{1}{n}}$$

If R - radius of convergence of $\sum_{n=0}^\infty c_n z^n$, then $\sum_{n=0}^\infty c_n z^n$ converges uniformly on any compact $K \subset \{z : |z| < R\}$ (on any D_r, $r < R$)

Prop

$$f(z) = \sum_{n=0}^\infty a_n z^n$$

$$g(z) = \sum_{n=0}^\infty b_n z^n$$

converge for all $|z| < R$. Then

1.) $f(z) + g(z) = \sum_{n=0}^\infty (a_n + b_n) z^n \quad \forall |z| < R$

2) $f'(z) = \sum_{n=1}^\infty n a_n z^{n-1}$ converges $\forall |z| < R$

$F := \sum_{n=0}^\infty \frac{a_n}{n+1} z^{n+1}$ converges $\forall |z| < R$

and $F'(z) = f(z)$

So, we can say $F(z) = \int_0^z f(\xi)\, d\xi$, $\lim n^{\frac{1}{n}} = 1$

3). $f(z) \cdot g(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) z^n \quad \forall |z| < R$.

<u>exercise</u> Try to prove this.

Thm Let $f : \underset{\mathbb{C}}{\underset{\cap}{\Omega}} \to \mathbb{C}$ such that $f'(z) = \lim_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$

exists $\forall z \in \Omega$. Then f is analytic in Ω

<u>Road map of the proof.</u>

1. If $f'(z)$ exists $\forall z \Rightarrow \int_\gamma f(z)\,dz = 0$, γ closed contour
2. $\Rightarrow f(z_0) = \dfrac{1}{2\pi i} \displaystyle\int_\gamma \dfrac{f(z)}{z - z_0}\,dz$
3. $\Rightarrow$ power series representation.

Complex Integration

Curve : $f(\langle a, b \rangle)$, f - continuous, injective function.

C^1 curve : $f(\langle a, b \rangle)$, $f \in C^1(\langle a, b \rangle)$ injective, $f'(t) \neq 0 \ \forall t$

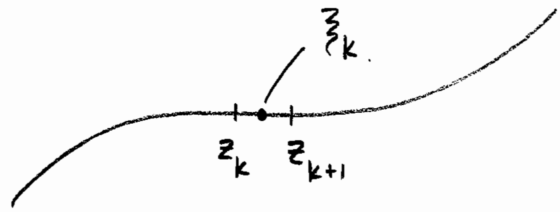
C^1 closed curve : $f(\text{circle})$

piecewise - C^1

$$\boxed{\begin{array}{l} \gamma - C^1 \text{ curve}, \quad \gamma = \varphi([a,b]) \\[1em] \displaystyle\int_\gamma f(z)\,dz := \int_a^b f(\varphi(t))\,\varphi'(t)\,dt. \end{array}}$$

Can define the complex integral as limit of Riemann sums

$\Sigma f(\xi_k)(z_{k+1}-z_k)\ d\xi$



$z = x+iy$

$dz = dx + i\,dy$

$f = U + iV$

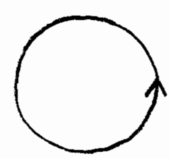
$\int_\gamma f(x,y)\,dx + i f(x,y)\,dy$

recall $\int_\gamma P(x,y)\,dx + Q(x,y)\,dy$

<u>2 ways to compute the complex integrations</u>

1 By parametrization

<u>example</u> $\displaystyle\int_{|z|=r} \frac{1}{z}\,dz$



$\varphi(t) = re^{it}$, $t \in [0, 2\pi]$

get $\displaystyle\int_{|z|=r} \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{\cancel{re^{it}}} \cdot \cancel{re^{it}} \cdot i\,dt$

$= 2\pi i$

<u>Observation</u> If $\exists F$ in a neighborhood of γ such that

$F'(z) = f(z)$ on γ, then $\int_\gamma f(z)\,dz = \underset{\varphi(b)}{F(\text{end})} - \underset{\varphi(a)}{F(\text{start})}$

<u>example</u> $\displaystyle\int_0^a z^n\,dz = \frac{z^{n+1}}{n+1}\Big|_0^a = \frac{a^{n+1}}{n+1}$

$n \geqslant 0$.