# COMPLEX FUNCTIONS §3

## Cauchy - Riemann equations.

Let $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ exists

$$\Delta z = \Delta x \Rightarrow \frac{\partial f}{\partial x}(z) = f'(z) \overset{\Delta z = i\Delta y}{=} -i \frac{\partial f}{\partial y}(z).$$

$$f'(z) \text{ exists} \Rightarrow \boxed{\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}}$$

let $\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right)$, $\overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)$

$$C\text{-}R \Leftrightarrow \overline{\partial} f = 0$$

**Proposition:**
$f$ is analytic then $\partial f = f'$

**Proof:**

$$\partial z = \frac{1}{2}(1 - ii) = 1$$

$$\partial(fg) = (\partial f)g + f(\partial g).$$

$$\Rightarrow \partial(z^2) = 2z \quad , \quad \partial(z^n) = nz^{n-1}$$

$$f(z) = \sum_{0}^{\infty} a_n (z - z_0)^n.$$

$$\Rightarrow \partial f(z) = \sum_{1}^{\infty} n a_n (z - z_0)^{n-1} = f'(z)$$

Another way: $f$ is analytic $\Rightarrow \frac{\partial f}{\partial x} = f'(z) = -i\frac{\partial f}{\partial y}.$

$$\Rightarrow f'(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial x}\right) = \partial f.$$

Remark: If $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are continuous (in open set $\Omega$)

then $f'(z)$ exists $\forall z \in \Omega$

Defn:

$f$ is differentiable at $x_0 \in R^n$ $f: R^n \to R^m$

if $f(x_0+h) = f(x_0) + Lh + \underbrace{o(|h|)}_{r(h)}$ where $\left|\frac{r(h)}{h}\right| \to 0$ as $h \to 0$

$$L_{jk} = \frac{\partial f_j}{\partial x_k}$$

$$f'(z) = a+ib \quad \left(= \begin{pmatrix} a \\ b \end{pmatrix} \text{ in vector form}\right)$$

$$\overset{\shortparallel}{\frac{\partial f}{\partial x}} = a+ib \qquad \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial x} = -b+ia \qquad \begin{pmatrix} -b \\ a \end{pmatrix}$$

$$\Rightarrow L = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Theorem: (Cauchy-Goursat)

Let R be a closed rectangle in C and $R \subseteq \Omega$ open

and let $f'(z)$ exists $\forall z \in \Omega$

Then $\int_{\partial R} f(z)\,dz = 0$

**Proof 1:** Assume $f'(z)$ continuous

or $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ continuous and satisfy C-R equs.

**Green's formula:**

**Stoke's theorem:**

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega \quad (*)$$

Apply $(*)$ for $\Omega = R$, $\omega = f(z)\,dz$

Observation

$$\left[\begin{array}{ll} \frac{\partial \cancel{f}}{\partial z} & \frac{\partial}{\partial \bar z} \\ f_z & f_{\bar z} \\ \partial f = f_z\,dz = \frac{\partial f}{\partial z}\,dz & \\ \bar\partial f = f_{\bar z}\,d\bar z & \end{array}\right]$$

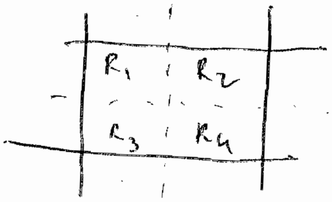
$$\begin{aligned} df &= f_x\,dx + f_y\,dy \\ &= \partial f + \bar\partial f \end{aligned}$$

$$\int_{\partial R} f(z)\,dz = \int_R (df) \wedge dz$$

$$= \int_R \Big(f_z\,dz + \underset{0 \text{ by C-R}}{\cancel{f_{\bar z}\,d\bar z}}\Big) \wedge dz = \int_R \frac{\partial f}{\partial z}\,dz \wedge dz$$

$$= 0.$$

**Proof 2:** $$\varphi(R) = \int_{\partial R} f(z)\,dz$$



$$\varphi(R) = \sum_{k=1}^{4} \varphi(R_k)$$

$\exists k$ st $|\varphi(R_k)| \geqslant \frac{1}{4} |\varphi(R)|$

Denote $R_k$ as $R^1$ $\quad |\varphi(R^1)| \geqslant \frac{1}{4} |\varphi(R)|$.

Similarly get $R^2$ st $\quad |\varphi(R^2)| \geqslant \frac{1}{4^2} |\varphi(R)|$.

$\vdots$

$R^n$ st $\quad |\varphi(R^n)| \geqslant \frac{1}{4^n} |\varphi(R)|$.

$$\bigcap_{n \geqslant 0} R^n = \{z_0\} \in R.$$

$\forall \varepsilon > 0, \exists \delta > 0$ st $\forall z$, $|z - z_0| < \delta \Rightarrow$ ~~$|f(z)-f(z_0)$~~

$\Rightarrow |f(z) - f(z_0) - f'(z_0)(z - z_0)| < \varepsilon |z - z_0|$.

Assume $\operatorname{diam} R^n < \delta$.

$$\int_{\partial R^n} f(z)dz = \int_{\partial R^n} \left[f(z) - f(z_0) - f'(z_0)(z - z_0)\right] dz.$$

$$|\varphi(R^n)| \leqslant 4 \operatorname{diam}(R^n) \cdot \varepsilon \operatorname{diam}(R^n)$$

$$= 4 \varepsilon (2^{-n} \operatorname{diam} R)^2$$

$\Rightarrow \quad |\varphi(R)| \leqslant 4 \varepsilon (\operatorname{diam} R)^2 \quad \forall \varepsilon > 0$

as $\quad |\varphi(R^n)| \geqslant \frac{1}{4^n} |\varphi(R)|$.

$\Rightarrow \varphi(R) = 0$. $\square$