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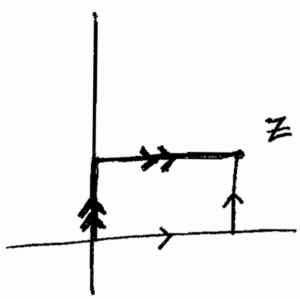
Last Time $\int_{\partial R} f(z)\,dz = 0$

Thm (Cauchy Theorem for the Disc)
Let $f'(z)$ exist $\forall z \in D$, where D is a disc. Then for all closed piecewise C^1 γ, $\int_\gamma f(z)\,dz = 0$.

Pf Recall from calculus:
$\int_\gamma P(x,y)\,dx + Q(x,y)\,dy = F(\text{end}) - F(\text{start})$ if $\exists F$ st $F_x = P$ and $F_y = Q$. This is proved using the fundamental theorem of calculus.
Idea of proof of Cauchy Thm: Construct an antiderivative.
Let a be the center of D.
$F(z) = \int_a^z f(\xi)\,d\xi$
$\frac{\partial F}{\partial x}(z) = f(z)$ and $\frac{\partial F}{\partial y}(z) = i f(z)$
We get this by taking $a = 0$.
$F(z) = \int_0^x f(s,0)\,ds + \int_0^y f(x,t)\,i\,dt$ where $\xi = s + it$.



We can use either curve and by Cauchy-G thm these coincide.

$\int f(z)\,dz = \int f(z)\,dx + i\int f(z)\,dy$
So F is an antiderivative of $f\,dz$. $\square$

Recall Forms that have antiderivatives are called exact.
($f_1 dx_1 + \cdots + f_n dx_n$ st $\exists F$ with $F_{x_k} = f_k$)
We will say a form is locally exact if there is a neighborhood where it is exact.

Observation One can integrate locally exact forms along any continuous path. (A path is a continuous mapping $\gamma : [a,b] \to X$.)
Idea: take the interval $[a,b]$ and partition it so that $\exists$ an antiderivative of ω in a neighborhood of $\gamma([a_k, a_{k+1}])$ for each such interval. We can determine the integral on the image of each of these small intervals.

Why is this well-defined?
Assume we have two different sets of points that partition $[a,b]$. Take the partition that uses all the points from both sets. Use this partition to determine the integral. But by different groupings we get the original two partitions, so all three integrals must be equal.

Lemma Let K be metric compact and let $\mathcal{U}$ be an open cover of K. $K \subset \bigcup_{U \in \mathcal{U}} U$.
Then $\exists r_0 > 0$ st $\forall r \le r_0$, $\forall x \in K$, $\exists U \in \mathcal{U}$ st $B(x,r) \subset U$.

Pf $\forall x \in K$ $\exists r(x)$ st $B(x, r(x)) \subset$ some $U \in \mathcal{U}$.
$\bigcup_{x \in K} B(x, \frac{r(x)}{2}) \supset K$. By compactness, $\exists x_1, \dots, x_n$ st
$\bigcup_1^N B(x_k, \frac{r_k}{2}) \supset K$. Define $r_0 = \min \frac{r(x_k)}{2}$.
Take $x \in K$. $\exists k$ st $x \in B(x_k, \frac{r(x_k)}{2})$
$B(x, r_0) \subset B(x_k, r(x_k)) \subset$ some $U \in \mathcal{U}$.