Lecture #5

$\boxed{\text{HW}}$ $z^n(1-z)^m$ Decompose in p. 123 #1 (b)

$\frac{1}{z^k}$ & $\frac{1}{(1-z)^j}$, integral $\neq 0$ only for $k=1, j=1$

$1 \leq k \leq n$ $\quad 1 \leq j \leq m$

$\boxed{\text{Remind}}$ $\int_\gamma f dz$ if $f'(z)$ exists $\forall z \in \Omega$ can be defined

for any path $\gamma : [a,b] \longrightarrow \Omega$

(True for any locally exact 1-form)

Given γ a path. $\boxed{\text{WLOG}}$ $\gamma : [0,1] \to \Omega$

$\boxed{\text{Def.}}$ $\gamma_1, \gamma_2 : [0,1] \longrightarrow \Omega$ two paths

γ_1 is <u>homotopy equivalent</u> to γ_2 (in Ω) if

$\exists$ cont. map $\phi : [0,1] \times [0,1] \longrightarrow \Omega$ s.t.

i) $\phi(x,0) = \gamma_1(x)$ $\quad \forall x \in [0,1]$

ii) $\phi(x,1) = \gamma_2(x)$ $\quad \forall x \in [0,1]$

Assume that either γ_1, γ_2 closed paths, i.e. $\gamma_1(1) = \gamma_1(0)$, $\gamma_2(1) = \gamma_2(0)$,

or $\gamma_2(0) = \gamma_1(0)$ & $\gamma_2(1) = \gamma_1(1)$

In this case in the def. we assume that

$\phi(0,t) \equiv \text{const.}$

$\phi(1,t) \equiv \text{const.}$

In the first case, assume

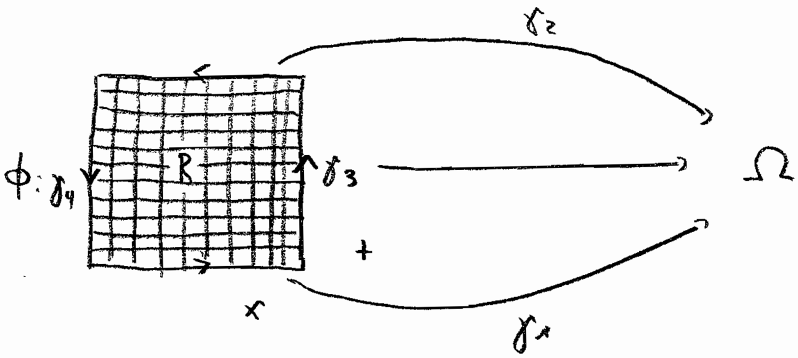
$\phi(0,t) = \phi(1,t) \quad \forall t \in I$

Thm. (Homotopy invariance of the integral)

Let $f: \underset{\text{open}}{\Omega} \longrightarrow \mathbb{C}$ s.t. $f'(z)$ exists $\forall z \in \Omega$. Let $\gamma_1 \sim \gamma_2$ in Ω.

Then
$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$$

Pf.



① $$\int_{\phi(\partial R)} f(z)\,dz = 0$$

$\phi|_{\partial R}$ is a closed path

$$\rightsquigarrow \int_{\phi(\partial R)} f(z)\,dz = \sum \int_{\phi(R_k)} f(z)\,dz$$

- Take R_k sufficiently small, s.t. $\forall R_k$ $\exists$ antiderivative of $f(z)dz$ in a nbhd. of $\phi(R_k)$
- Use Lebesgue number lemma

② $$\int_{\gamma_3} f(z)\,dz + \int_{\gamma_4} f(z)\,dz = 0$$

$$\overset{①}{\Rightarrow} \int_{\gamma_1} f(z)\,dz + \int_{-\gamma_2} f(z)\,dz = 0 \quad \square$$

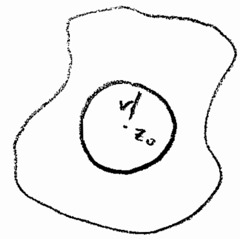
Thm. (Cauchy formula)

Let $f'(z)$ exist $\forall z$ in Ω & $\gamma \sim C(z_0, r)$ in $\Omega - \{z_0\}$

Moreover let $cl(D(z_0, r)) \subseteq \Omega$.

Then

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz = f(z_0)$$



Pf.

$$\int_\gamma \frac{f(z)}{z-z_0}\, dz = \int_{C(z_0, r)} \frac{f(z)}{z-z_0}\, dz$$

$\longleftarrow$ does not depend on r, if $cl(D(z_0,r)) \subseteq \Omega$

① $$\int_{C(z_0,r)} \frac{1}{z-z_0}\, dz = 2\pi i$$

② $$\int_{C(z_0,r)} \frac{f(z)}{z-z_0}\, dz = \int_{C(z_0,r)} \frac{f(z) - f(z_0)}{z-z_0}\, dz + \int_{C(z_0,r)} \frac{f(z_0)}{z-z_0}\, dz$$

$$=: I_1 + I_2$$

③ Then $I_2 = 2\pi i\, f(z_0)$ by ①

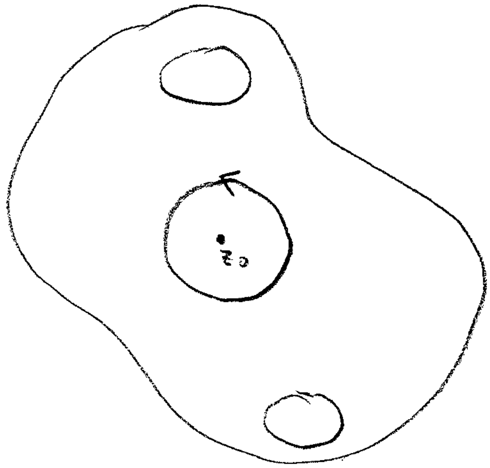
④ I_1 does not depend on $r > 0$ $\forall r < r_0$

$$|I_1| \leq 2\pi r \cdot \max_{|z-z_0| = r} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| \leq 2\pi r M \quad \forall r < \delta$$

with: $\lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists, so there ex. $\delta > 0$ s.t.

$\forall z,\ 0 < |z - z_0| < \delta \implies \left| \dfrac{f(z) - f(z_0)}{z - z_0} \right| \leq |f'(z_0)| + 1 =: M$

Therefore $I_1 = 0$


z_0