Recall from last time, we proved that:

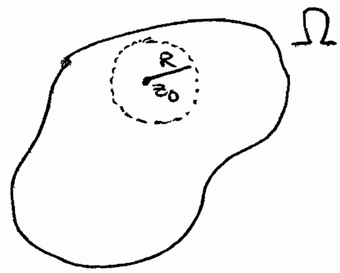
<u>Cauchy's Formula</u>/ If $f'(z)$ exists in an open set Ω, then

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz,$$ where γ is homotopic in $\Omega - \{z_0\}$ to a circle centered at z_0.

Our aim now is to show that:

<u>Thm</u>/ If $f'(z)$ exists $\forall z \in \Omega$, then f is analytic.

<u>Pf</u>/ Since Ω is open, $\forall z_0 \in \Omega$, there is an open disc $D(z_0; R) \subset \Omega$. We want to obtain a representation $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ for $|z-z_0| < R$. WLOG, we take $z_0 = 0$.
(In other words, set $w = z - z_0$.)



By Cauchy's Formula:

$$f(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(\xi)}{\xi - z}\,d\xi.$$

Let z be s.t. $|z| < R = |\xi|$. Then:

$$\frac{1}{\xi - z} = \frac{1}{\xi}\,\frac{1}{1-\frac{z}{\xi}} = \frac{1}{\xi}\sum_{n=0}^{\infty}\left(\frac{z}{\xi}\right)^n = \sum_{n=0}^{\infty}\frac{z^n}{\xi^{n+1}}$$

Since this is a geometric series, we have uniform convergence in z whenever $\left|\frac{z}{\xi}\right| < 1$ — i.e. $|z| < |\xi| = R$. Thus:

$$f(z) = \frac{1}{2\pi i}\int_{C_R} f(\xi)\frac{1}{\xi - z}\,d\xi \qquad C_R: |z| < R$$

$$= \frac{1}{2\pi i}\int_{C_R} f(\xi)\sum_{n=0}^{\infty}\frac{z^n}{\xi^{n+1}}\,d\xi$$

switch of sum and integral okay by uniform convergence.

$$= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\left(\int_{C_R}\frac{f(\xi)}{\xi^{n+1}}\,d\xi\right) z^n$$

$$= \sum_{n=0}^{\infty} a_n z^n, \quad \text{where } a_n = \frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{z^{n+1}}\,dz.$$

So, in general:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad \text{where } a_n = \frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

$\blacksquare$

Recall that if $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, then:

$$a_0 = f(z_0)$$

$$a_1 = f'(z_0)$$

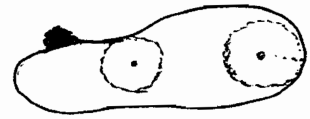
$$\vdots$$

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

Thus, as a byproduct of our proof, we have the formula:

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

Note that we've also learned something about the radius of convergence — it is essentially the distance to the boundary.



Laurent Series Representation

Let f be analytic in an annulus $A_{z_0; r, R}$

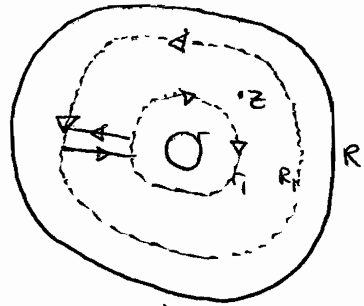
$$A_{z_0; r, R} = \{ z \in \mathbb{C} : r < |z-z_0| < R \}$$

Note that when $r=0$ we just have a punctured disc.

- So let's try to obtain a power series representation of f — but this time we'll need negative powers of n.

WLOG, take $z_0 = 0$, and consider the annulus $r < |z| < R$.

Let r_1, r_2, z be s.t. $r < r_1 < |z| < R_1 < R$.



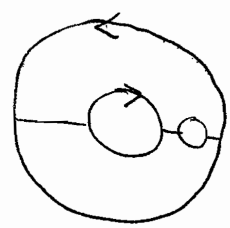
We claim that: $$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi,$$

where γ is the path shown in the diagram (Consisting of 2 circles with orientation given by the "left-leg rule.")

There are two ways to show this formula, depending on "whether you like homotopies or cutting stuff."

deform to circle around z

use a bridging path in the integration.



So $\gamma = \gamma_o \cup \gamma_i$ (outer and inner). Now:

$$\frac{1}{2\pi i}\int_{\gamma_0} \frac{f(\xi)}{\xi - z}\, d\xi = \sum_{n=0}^{\infty} a_n z^n,$$

where: $a_n = \frac{1}{2\pi i}\int_{\gamma_0} \frac{f(z)}{z^{n+1}}\, dz$. Also:

$$\frac{1}{2\pi i}\int_{\gamma_i} \frac{f(\xi)}{\xi - z}\, d\xi = \frac{1}{2\pi i}\int_{-\gamma_i} \frac{f(\xi)}{z - \xi}\, d\xi$$

$$= \frac{1}{2\pi i}\int_{-\gamma_i} f(\xi) \sum_{k=0}^{\infty} \frac{\xi^k}{z^{k+1}}\, d\xi$$

$n = k+1$

$$= \sum_{n=1}^{\infty} \frac{1}{2\pi i}\left(\int_{-\gamma_i} f(\xi)\, \xi^{n-1}\, d\xi\right) z^{-n}$$

$$= \sum_{n=1}^{\infty} b_n z^{-n},$$

where $b_n = \frac{1}{2\pi i}\int_{-\gamma_i} f(\xi)\, \xi^{n-1}\, d\xi$.

Now, the two curves γ_o and $-\gamma_i$ are homotopic (within the annulus),

so:
$$a_n = b_{-n} \qquad (n<0)$$

So, if Γ is any (closed) curve homotopic to γ_o (or $-\gamma_i$), then:

$$a_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi^{n+1}}\,d\xi$$

(counter-clockwise)

and so: $$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n.$$

This series expansion is called a <u>Laurent series</u>, and as this has shown, will work whenever f is analytic in an annulus.