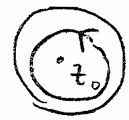
MATH 2250: Sep 17, 10

a) function f analytic in a disk centered at z_0



b) function f analytic in an annulus centered at z_0

$$f(z) = \sum a_n (z - z_0)^n$$

where $a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz$ where γ is a path $\sim$ a circle in the domain. /

In (a), $n \in \mathbb{Z}$, $n \geq 0$.

In (b), $n \in \mathbb{Z}$.

Let $f \in Hol(A_{1-\varepsilon, 1+\varepsilon}(0))$.

We can write $f(z) = \sum a_n z^n$ where $a_n = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z^{n+1}} dz$

$(z = e^{it})$ $$= \frac{1}{2\pi i} \int_0^{2\pi} f(e^{it}) e^{-i(n+1)t} \, i e^{it} dt$$

$$= \int_0^{2\pi} f(e^{it}) e^{-int} \frac{dt}{2\pi}$$ $\leftarrow$ Fourier's Serie.

Note: $\{e^{int}\}$ ONB in $L^2_{2\pi}(\frac{dt}{2\pi})$.

Liouville Theorem

Let $f \in Hol(\mathbb{C})$ (entire function), and let $|f(z)| < C$, $\forall z \in \mathbb{C}$.

Then $f(z) =$ constant.

PF $f(z) = \sum_{n \geq 0} a_n z^n$ where $a_n = \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{z^{n+1}} \, dz.$

and the series converges for all $z \in \mathbb{C}$.

Since all circles are homotopic, the integral does not depend on R.

$$|a_n| \leq \frac{1}{2\pi} \frac{C}{R^{n+1}} (2\pi R) \leq \frac{C}{R^n} \xrightarrow[n \geq 1.]{\infty} 0$$

Then $a_n = 0$, $\forall n \geq 1$.

Hence, $f(z) = a_0$. $\square$

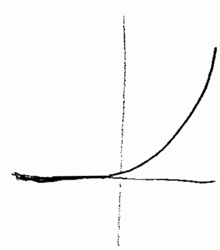
Uniqueness Theorem

Def Region - open and connected set.

Thm Let f be analytic in a region Ω, $z_0 \in \Omega$ s.t. $f^{(n)}(z_0) = 0$, $\forall n \geq 0$.
Then $f(z) = 0$, $\forall z \in \Omega$.

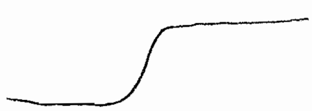
* This is not true for $C^\infty(\mathbb{R}^n)$.

ex $\varphi(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$, $\varphi^{(n)}(0) = 0$, $\forall n$.



$\varphi_1(x) = \varphi(x)\varphi(1-x)$

$\Phi(x) = \int_{-\infty}^{x} \varphi_1(t) \, dt$



Pf Use 'Continuous Induction'.

Let $U = \{ z \in \Omega : f^{(n)}(z) = 0, \ \forall n \geq 0 \}$.

$U_n = \{ z \in \Omega : f^{(n)}(z) = 0 \}$ is closed. (Because $f^{(n)}$ is continuous, $U_n = (f^{(n)})^{-1}(\{0\})$).

So $U = \bigcap_{n \geq 0} U_n$ is closed.

On the other hand, if $z_1 \in U$ then $\exists r > 0$ s.t. $\forall z, |z - z_1| < r$, $f(z) = \sum \frac{f^{(n)}(z_1)}{n!}(z - z_1)^n = 0$.

Thus, $\forall z \in B(z_1, r)$, $f^{(n)}(z) = 0$.

Hence, $B(z_1, r) \subset U$. That is U is open.

$U \neq \phi$ because $z_0 \in U$. Therefore, $U = \Omega$. $\square$

Thm 2 Let $f \in Hol(\Omega)$, Ω a region. Let $z_n \in \Omega$, $n \geq 0$, $\lim_{n \to \infty} z_n = z_0$. $f(z_n) = 0$, $\forall n$. and $z_n \neq z_0$, $\forall n \geq 1$.
Then $f(z) = 0$, $\forall z \in \Omega$.

Pf $f(z_0) = 0$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - 0}{z - z_0} = \lim_{n \to \infty} \frac{f(z_n)}{z_n - z_0} = 0.$$

So $f(z) = \sum_{n=2}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$.

Therefore,
$$f^{(2)}(z_0) = 2! \lim_{z \to z_0} \frac{f(z)}{(z - z_0)^2} = 2! \lim_{n \to \infty} \frac{f(z_n)}{(z_n - z_0)^2} = 0.$$

So $f(z) = \sum_{n=3}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$. Again $f^{(3)}(z_0) = 0$.

By induction, we have $f^{(n)}(z_0) = 0$, $\forall n$.

Apply the previous theorem. $\square$

Cor Zeroes of a non-zero analytic function in Ω cannot have accumulation point in Ω.

Pf If z_0 is an accumulation point of $\{z \in \Omega : f(z) = 0\}$, then $\exists z_n \neq z_0$, $n \geq 1$, $f(z_n) = 0$ s.t $\lim_{n \to \infty} z_n = z_0$.

Since f is continuous, then $f(z_0) = \lim f(z_n) = 0$. Contradiction. $\square$

Remark

Zeroes of $f \in Hol(\Omega)$ can have accumulation points in $\partial\Omega$.

ex $$\prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \cdot \frac{z - a_n}{1 - \bar{a}_n z}, \quad |a_n| < 1, \ a_n \neq 0.$$

converges iff $\sum (1 - |a_n|) < \infty$. for $|z| < 1$.

(Blaschke Product).