9/20 Complex Analysis

Classification of zeroes of singularities

$f \in Hol(\Omega)$ $z_0 \in \Omega$
$f(z_0) = 0$

1. all zeroes of f are isolated ($f \not\equiv 0$)
   - if $f(z_0) = 0$, then $\exists\, \delta$ st $\forall z$
   $0 < |z - z_0| < \delta \Rightarrow f(z) \neq 0$

~~Def~~ Order of zero
$f(z_0) = 0$
Order of z_0 = $\min\{n \mid f^{(n)}(z_0) \neq 0\}$
always positive
n = order of z_0 then $f(z) = (z - z_0)^n g(z)$
where $g(z_0) \neq 0$
since $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k = \sum_{k=n}^{\infty} (z - z_0)^k \frac{f^{(k)}(z_0)}{k!}$

Singularity - (Isolated Singularity)
$z_0 \in \Omega$, $f \in Hol(\Omega \setminus \{z_0\})$
then z_0 is an isolated singularity

1. Removable singularities
Thm Let $f \in Hol(\Omega \setminus \{z_0\})$ $z_0 \in \Omega$
Let $f(z)$ is bdd. in a neighborhood of z_0
$\exists\, \delta > 0, \exists M$ st $\forall z$ $0 < |z - z_0| < \delta \Rightarrow |f(z)| < M$ →

Then

1. $\lim_{z \to z_0} f(z)$ exists

2. $\tilde{f}(z) = \begin{cases} f(z) & z \neq z_0 \\ \lim_{z \to z_0} f(z) & z = z_0 \end{cases}$ is analytic

summarized by:

"f admits analytic continuation to all Ω"

Ex: $\frac{\sin z}{z}$

<u>Pf</u> $f(z) = \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n$ Laurent series

$$a_n = \int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^{n+1}} \frac{dz}{2\pi i} \quad \text{where } r < \delta$$

for $n \leq -1$

$$|a_n| \leq \underbrace{\frac{1}{2\pi} \cdot 2\pi R \cdot M \cdot R^{-1-n}}_{\substack{\downarrow R \to 0 \\ 0}} \qquad -1-n \geq 0$$

(since can choose any $R > 0$)

$\Rightarrow a_n = 0 \quad \forall n \leq -1$

Don't have any negative powers

$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n$ satisfies 1 and 2 //

2. <u>Pole</u>

Say $f(z) = \sum_{n=-N}^{\infty} a_n (z - z_0)^n$

Then z_0 is called a pole

Order of a pole $= \min \{ N \mid a_n = 0 \ \forall n < -N \}$

$= \min \{ N \mid (z - z_0)^N f(z)$ has removable singularity $\}$

$= \min\{N : f(z) = \frac{g(z)}{(z-z_0)^N} \quad g$ analytic in nbd. of z_0, $g(z_0) \neq 0\}$

Thm: z_0 is a pole iff $\lim_{z \to z_0} |f(z)| = \infty$

Pf: $\Rightarrow$ trivial

$\Leftarrow$ Let $\lim_{z \to z_0} |f(z)| = \infty$. Let $g(z) = \frac{1}{f(z)}$

$\Rightarrow \lim_{z \to z_0} |g(z)| = 0 \Rightarrow \lim_{z \to z_0} g(z) = 0$

$\Rightarrow$ g is bdd. in a nbd. of z_0

$\exists M < \infty, \delta > 0$ st $|g(z)| < M \quad \forall_{0 <} |z - z_0| < \delta$

$\Rightarrow z_0$ is a removable singularity for g, $g(z_0) = 0$, so $g(z) = (z-z_0)^N g_0(z)$, $g_0(z_0) \neq 0$.

$$\Rightarrow \quad f(z) = \frac{1/g_0(z)}{(z-z_0)^N} = \frac{f_0(z)}{(z-z_0)^N}$$

representation for a pole //.

3. Essential singularity

$f(z) = \sum_{n \in \mathbb{Z}} a_n (z-z_0)^n$ (inf. many negative powers)

$(\forall N > 0, \exists n < -N$ st $a_n \neq 0$)

(or $\operatorname{card}(n < 0 : a_n \neq 0) = \infty$)

Thm (Little Picard theorem)

If z_0 is essential singularity, then

$\forall\, \delta > 0$, $f(\underbrace{\{z \in \Omega \mid 0 < |z - z_0| < \delta\}}_{G})$ is dense

in $\mathbb{C}$

"Of course, Little is theorem, not Picard"

Pf Let $f(G)$ not dense in $\mathbb{C}$

Then ~~$\exists a \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$~~ $\exists a \in \mathbb{C}$

st $f(z) \notin$ a nbd. of $a = \{w \mid |w - a| < \varepsilon\}$

$\infty : \{w \mid |w| > R\}$

Then $\left| \dfrac{1}{f(z) - a} \right| \leq \dfrac{1}{\varepsilon}$

$\Rightarrow$ $g(z) := \dfrac{1}{f(z) - a}$ has removable singularity at z_0

$f(z) = \dfrac{1}{g(z)} + a$

$\Rightarrow$ z_0 is either a pole for f (if $g(z_0) = 0$)

or removable singularity $\lightning$ //